Exercise 10: The MI/SF transition in a gas of ultracold bosons.

(SF) $$|SF\rangle = \frac{1}{\sqrt{N!}}\left(\frac{1}{\sqrt{M}}\sum_i a_i^\dagger\right)^N |vac\rangle$$

$$\rho^{(1)}(i,j) = \langle SF|\, a_i^\dagger a_j\, |SF\rangle = \frac{N}{M} = 1$$

$$\langle n_i\rangle = \frac{N}{M} = 1$$

$$\langle n_i^2\rangle = \langle a_i^\dagger a_i a_i^\dagger a_i\rangle = \langle a_i^\dagger a_i^\dagger a_i a_i\rangle + \langle a_i^\dagger a_i\rangle =$$

$$= \frac{N(N-1)}{M^2} + \frac{N}{M} = \frac{N^2}{M^2} - \frac{N}{M^2} + \frac{N}{M} \simeq 2$$

$$\text{i.e. } g^{(2)}_{ii} = \langle a_i^\dagger a_i^\dagger a_i a_i\rangle = \frac{N^2}{M^2}\left(1-\frac{1}{N}\right) \simeq 1$$

$$\langle H\rangle = \left(-J\cdot C + \frac{U}{2}\right)\cdot N = \left(\frac{U}{2} - 6J\right)\cdot N$$

$\hookrightarrow$ coordination number $= 6$ in 3D.

only plane wave state with $\phi_i = 1/\sqrt{M}$ is occupied by N particles

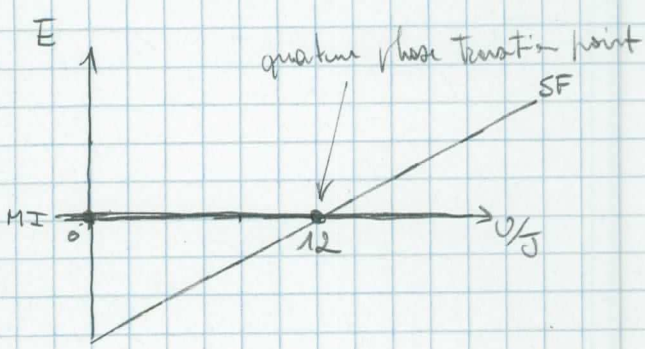
(MI) $$|MI\rangle = \left(\prod_i a_i^\dagger\right)|vac\rangle$$

$\rho^{(1)}(i,j) = \delta_{ij}$ $\longrightarrow$ all states occupied by 1 particle, both in real and momentum space.

$$\langle a_i^\dagger a_i^\dagger a_i a_i\rangle = 0, \text{ i.e. } \langle n_i^2\rangle = \langle n_i\rangle = \frac{N}{M} = 1$$

$$\Rightarrow \langle \Delta n_i^2\rangle = 0$$

$$\langle H\rangle = 0.$$



Change of lattice intensity:

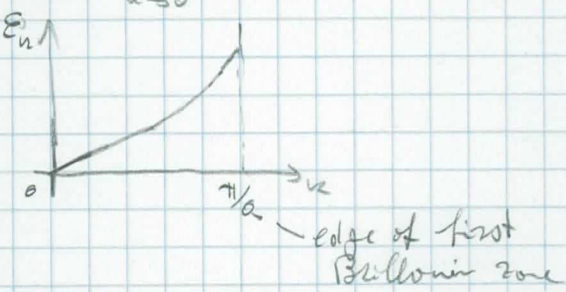
- tighter Wannier functions → increase U
- weaker hopping → reduce J

$\Longrightarrow$ net increase of U/J

<u>excitation spectrum</u>

(SF): excited states described by Bogoliubov theory.

$\Longrightarrow$ low energy Goldstone mode $\lim_{k \to 0} \mathcal{E}_k = 0$.



(MI): limit of vanishing J: lowest excitation contains 2 particles on a site and a hole in another site

$\Delta E = \frac{U}{2} 2 = U \longrightarrow$ gapped excitation

Momentum distribution

full Bose field $\quad \hat{\psi}(x) = \sum_i \phi(x-x_i)\, \hat{a}_i$

where $\phi(x-x_i)$ is Wannier function of i^{th} site.

$$\hat{\psi}(k) = \int dx\, e^{-ikx} \sum_i \phi(x-x_i)\, \hat{a}_i =$$

$$= \sum_i e^{-ikx_i}\, \hat{a}_i\, \tilde{\Phi}(k)$$

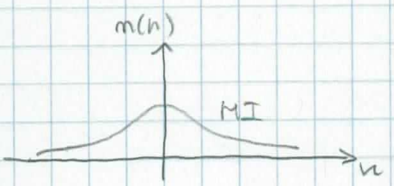
where $\tilde{\Phi}(k) = \int dx\, e^{-ikx}\, \phi(x)$

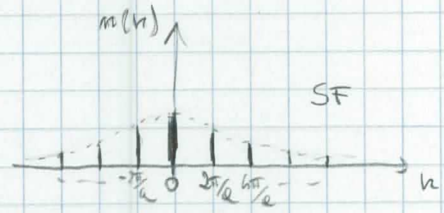
typical size of $\tilde{\Phi}(k)$ is $\Delta k \sim 1/\ell$,
ℓ being the size of W. function;
generally $\ell \ll a$, i.e. $\Delta k \gg \pi/a$

$$n(k) = \langle \psi^\dagger(k)\psi(k) \rangle = \sum_{ij} e^{ik(x_i - x_j)}\, |\tilde{\Phi}(k)|^2\, \langle a_i^\dagger a_j \rangle$$

(MI) $\quad n(k) = N\, |\tilde{\Phi}(k)|^2$

(SF) $\quad n(k) = |\tilde{\Phi}(k)|^2\, N^2\, \delta(k \bmod 2\pi/a)$





Momentum correlations in MI

$$\langle \psi^+(k)\, \psi^+(k')\, \psi(k')\, \psi(k) \rangle =$$

$$\sum_{\substack{i,j\\ i',j'}} e^{ik(x_i - x_j)}\, e^{ik'(x_{i'} - x_{j'})}\, |\tilde{\Phi}(k)|^2\, |\tilde{\Phi}(k')|^2 \,\cdot$$

$$\cdot \langle a_i^+ a_{i'}^+ a_{j'} a_j \rangle$$

$$\langle a_i^+ a_{i'}^+ a_{j'} a_j \rangle = \left(\delta_{ij}\, \delta_{i'j'} + \delta_{ij'}\, \delta_{i'j}\right)\left(1 - \delta_{ii'}\right)$$

$$= (N^2 - N)\, |\tilde{\Phi}(k)|^2 |\tilde{\Phi}(k')|^2 + \sum_{ij} e^{i(k-k')x_i}\, e^{-i(k-k')x_j}\, |\tilde{\Phi}(k)|^2 |\tilde{\Phi}(k')|^2$$

$$\cdot \left(1 - \delta_{ij}\right)$$

$$= |\tilde{\Phi}(k)|^2 |\tilde{\Phi}(k')|^2 \left[N^2 \left(1 - \frac{1}{N}\right) + N^2\, \delta\left((k-k') \bmod 2\pi/a\right) - N \right]$$

$$= |\tilde{\Phi}(k)|^2 |\tilde{\Phi}(k')|^2 \left[N^2 \left(1 + \delta\left((k-k') \bmod 2\pi/a\right)\right) - \cancel{2N} \right]$$

much smaller

$$= n(k)\, n(k') \left[1 + \delta\left((k-k') \bmod 2\pi/a\right) \right]$$

$\hookrightarrow$ positive correlations at $k - k' \equiv 0 \bmod 2\pi/a$

* shows different from interference effects in 2-well system in a Fock state.