exercise 7 : solution.

$$e^{i\hat{\phi}} = (\hat{a}^{+}\hat{a}+1)^{-1/2}\,\hat{a}$$

$$e^{i\phi}|n\rangle = \frac{1}{\sqrt{\hat{N}+1}}\cdot\hat{a}|n\rangle = \frac{1}{\sqrt{\hat{N}+1}}\sqrt{n}\cdot|n-1\rangle = |n-1\rangle$$

$$\left[e^{i\hat{\phi}},\hat{n}\right]|n\rangle = n|n-1\rangle - (n-1)|n-1\rangle = |n-1\rangle$$

$$\Longrightarrow \left[e^{i\phi},n\right] = e^{i\phi}$$

$$\left[e^{i\phi},e^{-i\phi}\right]|n\rangle = e^{i\phi}e^{-i\phi}|n\rangle - e^{-i\phi}e^{i\phi}|n\rangle = -\delta_{n,0}|0\rangle$$

$$\Longrightarrow \left[e^{i\phi},e^{-i\phi}\right] = -P_0 = -|0\rangle\langle 0|$$

$e^{i\phi}$ is NOT unitary because of vacuum state, i.e

$$e^{i\phi}e^{-i\phi} = \mathbb{1}, \qquad e^{-i\phi}e^{i\phi} = \mathbb{1}-P_0$$

$$\sin\phi = \frac{1}{2i}\left(e^{i\phi}-e^{-i\phi}\right), \qquad \left[\sin\phi,\hat{n}\right] = \frac{1}{2i}\left(e^{i\phi}+e^{-i\phi}\right) = \frac{1}{i}\cos\phi$$

$$e^{i\phi}|\psi\rangle = \varepsilon|\psi\rangle$$

$$\langle n|e^{i\phi}|\psi\rangle = \langle n+1|\psi\rangle = \psi_{n+1}$$

$$\Longrightarrow \psi_{n+1} = \varepsilon\psi_n \quad\Longrightarrow\quad \psi_n = (\varepsilon)^n\psi_0$$

$$|e^{i\phi}\rangle \longrightarrow \psi_n = e^{in\phi}$$

$$\int\frac{d\phi}{2\pi}|e^{i\phi}\rangle\langle e^{i\phi}| = \int\frac{d\phi}{2\pi}\sum_{nn'} e^{in\phi}|n\rangle\langle n'|e^{-in'\phi} = \sum_n |n\rangle\langle n| = \mathbb{1}$$

$$\psi(\phi) = \langle e^{i\phi}|\psi\rangle$$

$$|\psi\rangle = \int\frac{d\phi}{2\pi}\,|e^{i\phi}\rangle\langle e^{i\phi}|\psi\rangle = \int\frac{d\phi}{2\pi}\,\psi(\phi)\cdot|e^{i\phi}\rangle$$

univocally defines $|\psi\rangle$

$\psi(\phi) = \sum_n e^{-in\phi}\psi_n \implies \psi(\phi)$ only has "non-positive" Fourier components

is periodic in 2π.

basis are plane waves $\longleftrightarrow$ Fock states

coherent state $|coh:\alpha\rangle = \sum_n e^{-|\alpha|^2/2}\,\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle$

$$\psi_{|coh:\alpha\rangle}(\phi) = \langle e^{i\phi}|coh:\alpha\rangle = \sum_n e^{-in\phi}\,e^{-|\alpha|^2}\,\frac{\alpha^n}{\sqrt{n!}}$$

$$= e^{-|\alpha|^2}\sum_n|\alpha|^n\,\frac{e^{in(\theta_\alpha-\phi)}}{\sqrt{n!}}$$

$\frac{|\alpha|^n}{\sqrt{n!}}$ is max for $|\alpha|\simeq\sqrt{n}$

$$\log\frac{|\alpha|^n}{\sqrt{n!}} \simeq n\log|\alpha| - \frac{n}{2}\log n \simeq n_{max}\log|\alpha| - \frac{n_{max}}{2}\log n_{max} +$$

$$+\,\delta n\log|\alpha| - \frac{\delta n}{2}\log n_{max} - \frac{\cancel{n_{max}}}{2}\frac{\delta n}{\cancel{n_{max}}} + \frac{n_{max}}{4}\left(\frac{\delta n}{n_{max}}\right)^2$$

$$-\frac{\delta n}{2}\cdot\frac{\delta n}{n_{max}} + \dots$$

max for $\log|\alpha| = \frac{\delta n}{2}\left(\log n_{max}+1\right) \simeq \frac{\delta n}{2}\log n_{max}$.

width $\;-\frac{\delta n^2}{4n_{max}} \implies \Delta n \simeq \sqrt{2n_{max}} = \sqrt{2}\,|\alpha|$

$$\psi_{|coh:\alpha\rangle}(\phi) = C \cdot \int dn \; e^{in(\theta_\alpha - \phi)} \quad e^{-(n-n_{max})^2/4n_{max}}$$

$$\simeq e^{-(\theta_\alpha-\phi)^2 n_{max}}$$

$\Longrightarrow$ gaussian centered at θ_α, width $\dfrac{1}{\sqrt{2}\,|\alpha|} = \dfrac{1}{\sqrt{2}\,n_{max}}$

$$H = \frac{\hbar\chi}{2} n^2$$

$$i\hbar \frac{\partial}{\partial t}\psi(\theta) = i\hbar\frac{\partial}{\partial t}\langle e^{i\phi}|\psi(t)\rangle = \langle e^{i\phi}|H\psi\rangle$$

$$|\psi\rangle = \sum_n \psi_n |n\rangle \quad \Longrightarrow \quad H|\psi\rangle = \sum_n \frac{\hbar\chi}{2} n^2 \psi_n |n\rangle$$

$$i\hbar\frac{\partial}{\partial t}\psi(\theta) = \sum_n e^{-in\phi}\frac{\hbar\chi}{2} n^2 \psi_n = -\frac{\hbar\chi}{2}\frac{\partial^2}{\partial\phi^2}\psi(\phi)$$

$$i\hbar\frac{\partial}{\partial t}\psi(\phi) = -\frac{\hbar\chi}{2}\frac{\partial^2}{\partial\phi^2}\psi(\phi)$$

eigenstates $\psi_m(\phi) = e^{im\phi}$, $E_m = \dfrac{\hbar\chi}{2} m^2$

$\Longrightarrow$ number states

initial state : $|coh:\alpha\rangle \longrightarrow \psi(\phi) = e^{-(\theta_\alpha-\phi)^2|\alpha|^2}$

$$P \longrightarrow i\hbar\frac{\partial}{\partial\phi}, \quad m = \frac{\hbar}{\chi}, \quad X \rightarrow \phi$$

$$\Delta P \sim \hbar/\Delta x = \hbar\sqrt{2}\,|\alpha|$$

$$\Delta v \sim \frac{\Delta P}{m} = \frac{\hbar\sqrt{2}\cdot|\alpha|}{\hbar/\chi} = \sqrt{2}\,\chi\cdot|\alpha|$$

expands to $[0, 2\pi]$ within $t_{all} \sim \dfrac{1}{\chi|\alpha|} = \dfrac{1}{\chi|\alpha|^2}|\alpha|$

MF object

grows for $|\alpha| \to \infty$

periodic boundary conditions in ϕ, revival possible:

$$E_n = \frac{\hbar \chi}{2} n^2 \qquad \text{for} \quad t_{rev} = \frac{2\pi}{\chi/2}$$

$$\Rightarrow \frac{1}{\hbar} E_n \cdot t_{rev} = \frac{1}{\cancel{\hbar}} \frac{\cancel{\hbar}\chi}{\cancel{2}} n^2 \frac{2\pi}{\chi/2} = 2\pi n^2$$

integer multiple of 2π

$\Rightarrow$ revival of the phase

at $t_{rev}/2$:

$$|\psi(t_{rev}/2)\rangle = \sum_n e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} e^{-i\frac{\chi}{2} n^2 t_{rev}/2} |n\rangle =$$

$$= \sum_n e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} (-1)^n |n\rangle = |coh: -\alpha\rangle$$

at $t_{rev}/4$:

$$|\psi(t_{rev}/4)\rangle = \sum_n e^{-|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} e^{-i\frac{\pi}{2} n^2} \quad \begin{cases} 1 & \text{for } n \text{ even} \\ -i & \text{for } n \text{ odd} \end{cases}$$

$$A|\alpha\rangle + B|-\alpha\rangle = \sum_n e^{-|\alpha|^2/2} \frac{A\alpha^n + B(-\alpha)^n}{\sqrt{n!}} |n\rangle =$$

$$= \sum_n e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} (A + (-1)^n B) |n\rangle$$

$$A + B = 1 \quad , \quad A - B = -i$$

$$\Rightarrow A = \frac{1-i}{2} \quad , \quad B = \frac{1+i}{2}$$

$$\Rightarrow |\psi(t_{rev}/4)\rangle = A|\alpha\rangle + B|-\alpha\rangle$$ <u>Schrödinger cat</u>

$t_{rev} = \frac{4\pi}{\chi} = \frac{4\pi}{\chi|\alpha|^2} \cdot |\alpha|^2$ grows as $|\alpha|^2$, while $t_{coll} \sim |\alpha|$