σ-lis

<u>example of diagonalization of Bogoliubov Hamiltonian</u>

homogeneous system at rest $\quad \phi_0(r) = \phi_0 \;, \quad \phi_0 \in \mathbb{R}$

$$H_{Bog} = \frac{1}{2} (\Lambda^\dagger, \Lambda) \, \eta \, \mathcal{L} \begin{pmatrix} \Lambda \\ \Lambda^\dagger \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} -\frac{\hbar^2}{2m}\nabla^2 + \mu_0 \,, & \mu_0 \\ -\mu_0 & +\frac{\hbar^2}{2m}\nabla^2 - \mu_0 \end{pmatrix}$$

<u>translational invariance</u> $\longrightarrow$ eigenvectors $\begin{pmatrix} U \\ V \end{pmatrix} e^{ikr}$

<u>for each $\underline{k}$</u>

$$\vec{v}_{+,k} = \begin{pmatrix} U_k \\ V_k \end{pmatrix} e^{ikr} \;, \quad \varepsilon_k = \sqrt{\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + 2\mu_0\right)}$$

$$U_k \pm V_k = \left(\frac{\hbar^2 k^2 / 2m}{\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + 2\mu_0\right)} \right)^{\pm 1/2}$$

$$\vec{v}_{-,k} = \begin{pmatrix} V_k \\ U_k \end{pmatrix} e^{ikr} \;, \quad \varepsilon_k = -\sqrt{\quad \dots}$$

same U_k, V_k

$$\int \vec{v}^{\,\dagger}_{+,k} \, \eta \, \vec{v}_{+,k'} = (2\pi)^d \delta(\underline{k} - \underline{k}')$$

$$\int \vec{v}^{\,\dagger}_{-,k} \, \eta \, \vec{v}_{+,k'} = 0$$

$$\int \vec{v}^{\,\dagger}_{-,k} \, \eta \, \vec{v}_{-,k'} = -(2\pi)^d \delta(\underline{k} - \underline{k}')$$

decompose $\begin{pmatrix} \Delta \\ \Delta^+ \end{pmatrix}$ over eigenvectors of $\mathcal{L}$:

$$\hat{b}_k = \vec{v}^{\,+}_{+,k} \, \eta \begin{pmatrix} \Delta \\ \Delta^+ \end{pmatrix}$$

$$\hat{c}_k = v^+_{-,k} \, \eta \begin{pmatrix} \Delta \\ \Delta^+ \end{pmatrix} = - \hat{b}^+_{-k}$$

$$\begin{pmatrix} \Delta \\ \Delta^+ \end{pmatrix} = \int \frac{dk}{(2\pi)^d} \, \vec{v}_{+,k} \cdot \hat{b}_k - \vec{v}_{-,k} \cdot \hat{c}_k =$$

$$= \int \frac{dk}{(2\pi)^d} \begin{pmatrix} U_k \\ V_k \end{pmatrix} e^{ikx} \, \hat{b}_k + \begin{pmatrix} V_k \\ U_k \end{pmatrix} e^{-ikx} \, \hat{b}^+_k$$

$$\mathcal{L} \begin{pmatrix} \Delta \\ \Delta^+ \end{pmatrix} = \int \frac{dk}{(2\pi)^d} \, \omega^{Bog}_k \begin{pmatrix} U_k \\ V_k \end{pmatrix} e^{ikx} \, \hat{b}_k - \omega^{Bog}_k \begin{pmatrix} V_k \\ U_k \end{pmatrix} e^{-ikx} \, \hat{b}^+_k$$

$$\frac{1}{2} (\Delta^+ \Delta) \, \eta \, \mathcal{L} \begin{pmatrix} \Delta \\ \Delta^+ \end{pmatrix} = \frac{1}{2} \int \frac{dk}{(2\pi)^d} \, \omega^{Bog}_k \left(\hat{b}^+_k \hat{b}_k + \hat{b}_k \hat{b}^+_k \right) =$$

$$= c^{te} + \int \frac{dk}{(2\pi)^d} \, \hat{b}^+_k \hat{b}_k \, \omega^{Bog}_k$$

Physical meaning of squeezed form of Bogoliubov vacuum

$$\hat{\rho}(x) = \psi^+(x)\,\psi(x) \cong |\phi(x)|^2 \left(N - \int dy\; \Lambda^+(y)\Lambda(y)\right) +$$

$$+ \sqrt{N}\left(\phi_0(x)\,\Lambda^+(x) + \phi_0^*(x)\,\Lambda(x)\right) + \Lambda^+(x)\,\Lambda(x)$$

(homogeneous system)
$$= \frac{N}{V} - \frac{1}{V}\int dy\; \Lambda^+(y)\Lambda(y) + \Lambda^+(x)\Lambda(x) + \sqrt{\frac{N}{V}}\left(\Lambda^+(x) + \Lambda(x)\right)$$

$$\hat{\rho}(k\neq 0) = \int \frac{dx}{\sqrt{V}}\, e^{-ikx}\,\hat{\rho}(x) = \int\frac{dx}{\sqrt{V}}\, e^{-ikx}\left(\Lambda^+(x)\Lambda(x) + \sqrt{\frac{N}{V}}\left(\Lambda^+(x) + \Lambda(x)\right)\right)$$

$$= \sqrt{n}\int\frac{dx}{\sqrt{V}}\, e^{-ikx}\sum_{k'}\left(\frac{U_{k'}}{\sqrt{V}}\, e^{ik'x}\, b_{k'} + \frac{V_{k'}}{\sqrt{V}}\, e^{-ik'x}\, b_{k'}^+ + h.c.\right)$$

$$+ \text{higher order terms}$$

$$= \sqrt{n}\left[U_k\, b_k + V_{-k}\, b_{-k}^+ + h.c.\right] = \sqrt{n}\,(U_k + V_k)\,(b_k + b_{-k}^+)$$

Analogously
$$\hat{j}(x) = -\frac{i\hbar}{2m}\left(\psi^+\cdot\nabla\psi - \nabla\psi^+\cdot\psi\right)$$

$$\Rightarrow\quad \hat{j}(k) = \frac{\sqrt{n}\,\hbar k}{2m}\,(U_k - V_k)\,(b_k - b_{-k}^+) = \frac{\hbar k}{2m}\sqrt{n}\,\left(\Lambda_k - \Lambda_{-k}^+\right)$$

Under free evolution:

$$\frac{d}{dt}\rho(k) = \sqrt{n}\,(U_k + V_k)\cdot(-i\omega_k\, b_k + i\omega_{-k}\, b_{-k}^+) =$$

$$= -i\,\omega_k\sqrt{n}\,(U_k + V_k)\,(b_k - b_{-k}^+) =$$

and $\operatorname{div}\vec{j} \longrightarrow$
$$ik\cdot j(k) = i\frac{\hbar k^2}{2m}\sqrt{n}\,(U_k - V_k)\,(b_k - b_{-k}^+)$$

So:
$$\frac{d}{dt}\rho(k) + ik\cdot j(k) = -i\sqrt{n}\left[(U_k + V_k)\,\omega_k - \frac{\hbar k^2}{2m}(U_k - V_k)\right] = \emptyset$$

$$\langle \varrho^+(k)\,\varrho(k)\rangle = n\,(U_k+V_k)^2 \langle (b_k^+ + b_{-k})(b_k + b_{-k}^+)\rangle$$

$$= n\,(U_k+V_k)^2 \left(\frac{2}{e^{\beta\omega_k}-1}+1\right) = n\,\frac{\hbar^2k^2/2m}{\omega_k}\left(\frac{2}{e^{\beta\omega_k}-1}+1\right)$$

$$\langle j^+(k)\,j(k)\rangle = n\,\frac{\hbar^2k^2}{4m^2}\,\frac{\omega_k}{\hbar^2k^2/2m}\left(\frac{2}{e^{\beta\omega_k}-1}+1\right) =$$

$$= \frac{n\,\omega_k}{2m}\left(\frac{2}{e^{\beta\omega_k}-1}+1\right)$$

NOTE:

$$\langle \varrho(k)\,\varrho(-k)\rangle = n\,(U_k+V_k)^2 \cdot \langle (b_k^+ + b_{-k})(b_{-k}^+ + b_k)\rangle = \langle \varrho^+(k)\,\varrho(k)\rangle$$

for low k:

$$\langle \varrho^+(k)\,\varrho(k)\rangle \longrightarrow n\,\frac{k_BT}{\omega_k^2}\,\frac{\hbar^2k^2}{m}$$

$$\langle j^+(k)\,j(k)\rangle \longrightarrow \frac{n\,k_BT}{m}$$

* At a given point x

$$\langle \varrho(x)\cdot\varrho(x)\rangle - \langle\varrho(x)\rangle^2 = \langle \sqrt{n}^{\,2}\,(\Lambda(x)+\Lambda^+(x))^2\rangle =$$

$$= n\,\langle (\Lambda(x)+\Lambda^+(x))^2\rangle =$$

$$= \langle n\cdot\sum_{kk'}\frac{(U_k+V_k)}{\sqrt{V}}(b_k+b_{-k}^+)\cdot\left(\frac{U_{k'}+V_{k'}}{\sqrt{V}}\right)(b_{k'}+b_{-k'}^+)\rangle =$$

$$= \frac{n}{V}\cdot\sum_k (U_k+V_k)^2\langle (b_k+b_{-k}^+)^2\rangle = \frac{n}{V}\sum_k (U_k+V_k)^2\left(\frac{2}{e^{\beta\omega_k}-1}+1\right)$$

$$\langle j(x)\,j(x)\rangle = \frac{n}{V}\sum_k\left(\frac{\hbar k}{2m}\right)^2(U_k-V_k)^2\langle (b_k-b_{-k}^+)^2\rangle =$$

$$= \frac{n}{V}\sum_k\left(\frac{\hbar k}{2m}\right)^2(U_k-V_k)^2\langle (b_k-b_{-k}^+)^2\rangle$$

single particle $\longrightarrow$ $\langle j(x)\, j(x) \rangle$ has a factor $\left(\frac{\hbar k}{2m}\right)^2$ more than $\langle \delta\rho(x)\, \delta\rho(x) \rangle$

interactions $\longrightarrow$ $\begin{cases} \text{enhance } \langle j(x)\, j(x)\rangle \text{ by } (U_k - V_k)^2 \\ \text{suppress } \langle \delta\rho(x)\, \delta\rho(x)\rangle \text{ by } (U_k + V_k)^2 \end{cases}$

$$\Downarrow$$

SQUEEZING effect

NOTE: Heisenberg is maintained as $(U_k + V_k)(U_k - V_k) = 1$.

NOTE 2: $\underbrace{\frac{1}{\sqrt{2}}\left(b_k + b_{-k}^{+}\right)}_{= X}$ and $\underbrace{\frac{-i}{\sqrt{2}}\left(b_{-k} - b_k^{+}\right)}_{= P}$

are "conjugate variables" in 2-mode squeezing

$\longrightarrow [X, P] = i$

see e.g. D.F. Walls and G.J. Milburn, "Quantum Optics", Springer, Berlin, 1994