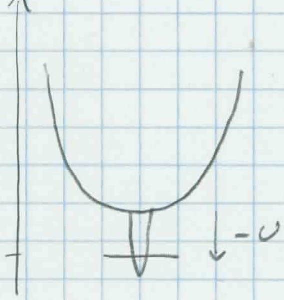
Exercise 3 : reversible formation of a BEC

Harmonic trap :

$$N_{nc} = \left(\frac{k_B T}{\hbar\omega}\right)^3 g_3(e^{\beta\mu})$$



+ narrow dip : single level at $-U$

$\longrightarrow$ μ is limited to $\mu < -U$

$\longrightarrow$ no condensate in harmonic trap.

$\longrightarrow$ population in single level : $N_0 = \dfrac{1}{e^{\beta\mu} - 1}$

Dip is switched on adiabatically $\longrightarrow$ total S is conserved

harmonic trap $\longrightarrow$ $S = k_B N_{nc} \left[4 \dfrac{g_4(e^{\beta\mu})}{g_3(e^{\beta\mu})} - \beta\mu \right]$

no entropy associated to discrete level.
(vanishes in T.D. limit)

initially : no BEC , $\mu_i < 0$, $N_{nc} = N$

$$S_i = k_B N \left[\frac{4 g_4(e^{\beta_i\mu_i})}{g_3(e^{\beta_i\mu_i})} - \beta_i\mu_i \right], \quad N = (\beta_i\hbar\omega)^{-3} g_3(e^{\beta\mu_i})$$

finally : BEC in discrete level , $\mu_f = -U$

$$S_f = k_B N_{mc} \left[\frac{4 g_4(e^{-\beta_f U})}{g_3(e^{-\beta_f U})} + \beta_f U \right]$$

$$N_{mc} = \frac{1}{(\beta_f \hbar \omega)^3} \; g_3(e^{-\beta_f U})$$

Simplest case : $\mu_i = 0$, on the edge of BEC

$$S_i = k_B N \left[4 \frac{g_4(1)}{g_3(1)} \right]$$

Entropy of a BEC

isolated BEC at a given N $\longrightarrow$ no entropy

single mode thermally occupied:

$$S = \left[(1+\bar{n}) \log (1+\bar{n}) - \bar{n} \log \bar{n} \right]$$

for large $\bar{n}$ $\quad S = \left[\log(1+\bar{n}) + \bar{n} \log(1+\bar{n}) - \bar{n} \log \bar{n} \right] =$

$$= \left[\log \bar{n} + \log\left(1+\frac{1}{\bar{n}}\right) + \bar{n} \cancel{\log \bar{n}} + \bar{n} \log\left(1+\frac{1}{\bar{n}}\right) + \right.$$

$$\left. - \bar{n} \cancel{\log \bar{n}} \right] =$$

$$\simeq \left[\log \bar{n} + \frac{1}{\bar{n}} + \bar{n} \cdot \frac{1}{\bar{n}} + O\left(\frac{1}{\bar{n}}\right) \right] =$$

$$= \log \bar{n} + 1 + O\left(\frac{1}{\bar{n}}\right)$$

Arguments to neglect it:

* scales less than extensively:

$$\bar{n} \sim N_{TOT} \quad , \quad \frac{S}{N} \sim \frac{\log \bar{n}}{\bar{n}} \longrightarrow 0 \quad \text{as } N_{TOT} \to \infty.$$

* within canonical ensemble N_0 fixed by $N_0 = N - N_0$ so it does not carry entropy by itself

To prove expression for S:

$$S = - \frac{d\Omega}{dT} = - \frac{d}{dT} \left[k_B T \log Z \right] = - \frac{d}{dT} \left[k_B T \log \left(1 - e^{\beta(\mu - \epsilon_i)}\right) \right.$$

$$= k_B \log\left(\frac{1}{1-e^{\beta(\mu-\varepsilon_i)}}\right) - k_B T \frac{(-)e^{\beta(\mu-\varepsilon_i)} \cdot (\mu-\varepsilon_i)\left(-\frac{1}{k_B T^2}\right)}{1-e^{\beta(\mu-\varepsilon_i)}}$$

$$\bar{n} = \frac{1}{e^{\beta(\varepsilon_i-\mu)}-1} = \frac{e^{\beta(\mu-\varepsilon_i)}}{1-e^{\beta(\mu-\varepsilon_i)}} = -\frac{1-e^{\beta(\mu-\varepsilon_i)}}{1-e^{\beta(\mu-\varepsilon_i)}} + \frac{1}{1-e^{\beta(\mu-\varepsilon_i)}} =$$

$$\frac{1}{1-e^{\beta(\mu-\varepsilon_i)}} = \bar{n}+1$$

$$S = k_B \log(1+\bar{n}) + k_B T \frac{1}{e^{\beta(\varepsilon_i-\mu)}-1} \frac{(\mu-\varepsilon_i)}{(-k_B T^2)}$$

$$= k_B \left[\log(1+\bar{n}) - \bar{n}\left(\frac{\mu-\varepsilon_i}{k_B T}\right)\right] =$$

$$\bar{n} = \frac{1}{e^{\beta(\varepsilon_i-\mu)}-1} \qquad e^{\beta(\varepsilon_i-\mu)} = 1+\frac{1}{\bar{n}}$$

$$\frac{\mu-\varepsilon_i}{k_B T} = -\log\left(1+\frac{1}{\bar{n}}\right)$$

$$S = k_B\left[\log(1+\bar{n}) + \bar{n}\log\left(1+\frac{1}{\bar{n}}\right)\right] =$$

$$= k_B\left[(1+\bar{n})\log(1+\bar{n}) + \bar{n}\log\left(1+\frac{1}{\bar{n}}\right) - \bar{n}\log(1+\bar{n})\right] =$$

$$= k_B\left[(1+\bar{n})\log(1+\bar{n}) + \bar{n}\left[\log\left(\frac{\cancel{\bar{n}+1}}{\bar{n}} \Big/ \cancel{1+\bar{n}}\right)\right]\right] =$$

$$= k_B\left[(1+\bar{n})\log(1+\bar{n}) - \bar{n}\log\bar{n}\right]$$