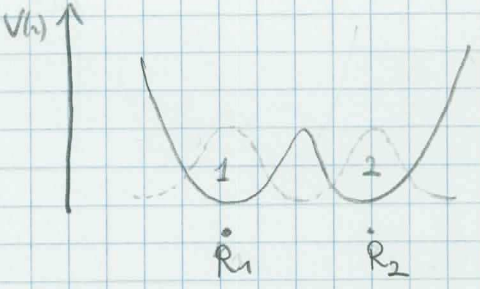
Lecture 5 : the phase of a BEC

Condensate in 2 well geometry



e.g. $\psi_{1,2}(r) = e^{-r^2/2\sigma^2}$, T.F. ...

GPE description : $\psi(r) \simeq \psi_1(r-R_1) + \psi_2(r-R_2)$

* same phase → avoid nodes of wf

⟹ 2 coherent BECs

After expansion (for simplicity neglect interactions) :

$$\psi(r, \text{large } t) \simeq \tilde{\psi}\left(k = \frac{mr}{\hbar t}, t=0\right)$$

* time-of-flight ballistic expansion

$\frac{\hbar k}{m} \cdot t = vt = r \gg \Delta r_0$ initial size of BECs.

$$= \tilde{\psi}_1(k)\, e^{ikR_1} + \tilde{\psi}_2(k)\, e^{ikR_2}$$

Density profile after expansion :

$$|\psi(r, \text{large } t)|^2 \simeq |\tilde{\psi}_{1,2}(k)|^2 \cos^2\left(\frac{k\cdot(R_1-R_2)}{2}\right)$$

assume identical : envelope function

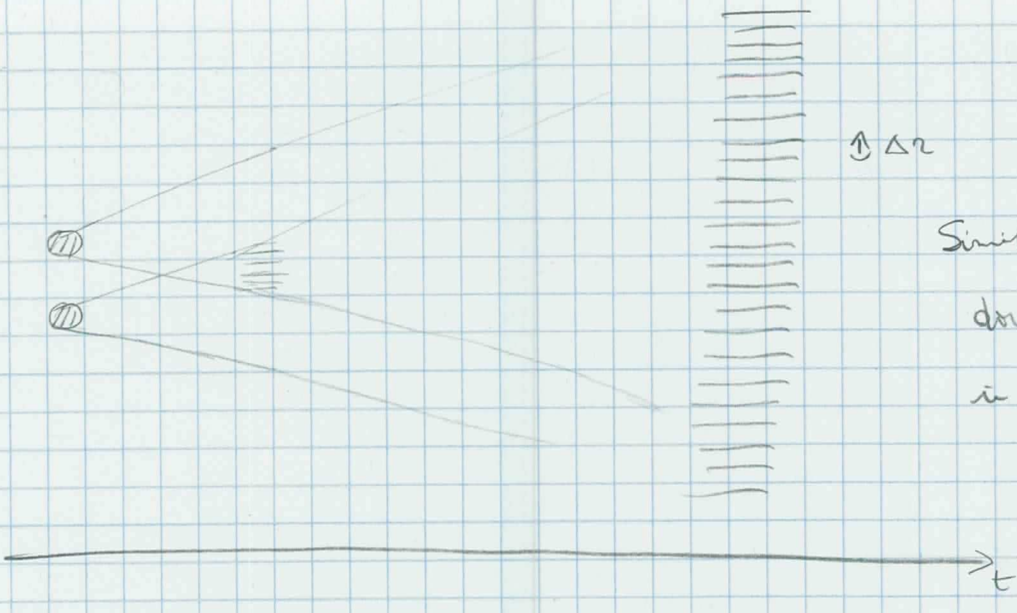
→ oscillations.

Oscillating Terms : $\cos^2\left(k\cdot(R_1-R_2)/2\right) = \frac{1}{2}\left[1 + \cos\left[\frac{m}{\hbar t}\, r\cdot(R_1-R_2)\right]\right]$

if $d = |r_1 - r_2|$:

period of oscillations $\frac{m}{\hbar t} \Delta r \cdot d = 2\pi$

$$\Rightarrow \Delta r = \frac{2\pi \hbar t}{m d}$$



Similar to Young double-slit exp. in far field.

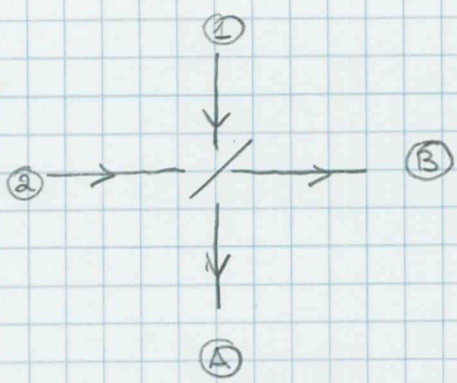
NOTE: Ketterle's experiment was performed in a slightly different geometry where fringes were observed within the BECs.

* interactions drive expansion
* fringes observed between the BECs.

If condensates have non-vanishing relative phase θ :

$\rightarrow$ fringes shifted by θ

Simplified scheme:



$$\text{in} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \xrightarrow{S} \text{out} \begin{Bmatrix} A \\ B \end{Bmatrix}$$

$S \rightarrow$ scattering matrix.

50/50 semi-transmitting plate

$$S_\vartheta = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\vartheta} \\ -e^{-i\vartheta} & 1 \end{pmatrix}$$

$$\begin{pmatrix} \hat{a}_A \\ \hat{a}_B \end{pmatrix} = S_\vartheta \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

unitarity $\quad S_\vartheta^\dagger S_\vartheta = \mathbb{1}$

$\longrightarrow \hat{I}_A + \hat{I}_B = \hat{I}_1 + \hat{I}_2$.

$$\hat{I}_A = \hat{a}_A^\dagger \hat{a}_A = \frac{1}{2}\left(\hat{a}_1^\dagger + e^{-i\vartheta}\hat{a}_2^\dagger\right)\left(\hat{a}_1 + e^{i\vartheta}\hat{a}_2\right) =$$

$$= \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + e^{i\vartheta}\hat{a}_1^\dagger \hat{a}_2 + e^{-i\vartheta}\hat{a}_2^\dagger \hat{a}_1\right)$$

example 1: $\quad$ coherent initial states $\quad \langle \hat{a}_{1,2} \rangle = \alpha_{1,2}$

$$\alpha_{1,2} = \alpha_0 \, e^{\pm i\phi/2}$$

$$\langle \hat{I}_A \rangle = \frac{1}{2}\left[|\alpha_0|^2 + |\alpha_0|^2 + e^{i\vartheta}e^{-i\phi}|\alpha_0|^2 + e^{-i\vartheta}e^{i\phi}|\alpha_0|^2\right] =$$

$$= |\alpha_0|^2 \left[1 + \cos(\vartheta - \phi)\right]$$

$\hookrightarrow$ fringes

example 2: $\quad$ Fock states $\quad |1 : N/2 \,;\, 2 : N/2\rangle = \dfrac{(a_1^\dagger)^{N/2}}{\sqrt{(N/2)!}} \dfrac{(a_2^\dagger)^{N/2}}{\sqrt{(N/2)!}} |vac\rangle$

phase totally random $\quad (\Delta N \, \Delta\phi \geq 1)$

in fact:

$$\langle \hat{I}_A \rangle = \frac{1}{2}\left[N/2 + N/2 + \emptyset + \emptyset \right] = N$$

with no fringes.

Questions:

(i) what is probability of all N particles in A ?

(ii) Ketterle's experiment with Fock states ?

(i) If particles distinguishable : $P(A:N) = (1/2)^N$

Quantum statistics effect: $|\psi\rangle = \frac{1}{(N/2)!} (a_1^+)^{N/2} (a_2^+)^{N/2} |vac\rangle$

for $\theta = 0$:

$$\hat{a}_1 = \frac{1}{\sqrt{2}}(\hat{a}_A + \hat{a}_B)$$

$$\hat{a}_2 = \frac{1}{\sqrt{2}}(\hat{a}_A - \hat{a}_B)$$

$$|\psi\rangle = \frac{1}{(N/2)!}\frac{1}{2^{N/2}} (a_A^+ + a_B^+)^{N/2} (a_A^+ - a_B^+)^{N/2} |vac\rangle =$$

$$= \frac{1}{(N/2)!}\frac{1}{2^{N/2}} \left(a_A^{+2} - a_B^{+2}\right)^{N/2} |vac\rangle =$$

$$= \cancel{\frac{1}{(N/2)!}} \frac{1}{2^{N/2}} \sum_{i=0}^{N/2} (-1)^i (a_A^+)^{2i} (a_B^+)^{N-2i} |vac\rangle \frac{\cancel{(N/2)!}}{i!\,(N/2-i)!}$$

$$= \frac{1}{2^{N/2}} \sum_{i=0}^{N/2} \frac{(-1)^i}{i!\,(\frac{N}{2}-i)!} (a_A^+)^{2i} (a_B^+)^{N-2i} |vac\rangle$$

$$P(A:N) = \left(\frac{1}{2^{N/2}} \frac{1}{(N/2)!} \sqrt{N!}\right)^2 = \frac{N!}{2^N}\frac{1}{((N/2)!)^2} \simeq \frac{\left(\frac{N}{e}\right)^N \sqrt{2\pi N}}{\cancel{2^N}\left(\left(\frac{N}{\cancel{2}e}\right)^{N/2}\sqrt{2\pi N/2}\right)^2} \simeq \sqrt{\frac{2}{\pi N}} \gg \frac{1}{2^N}$$

[have used Stirling's formula $n! \simeq \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$]

$\Rightarrow$ even Fock states keep significant high-order correlations

(ii) Take 2 condensates and measure $\hat{N}_1$ and $\hat{N}_2$ exactly

$\longrightarrow$ BECs in Fock states. No info on phase possible

$$\hat{\psi}_{out}(r) \simeq \hat{\psi}_1(k)\, e^{ikR_1} + \hat{\psi}_2(k)\, e^{ikR_2} \simeq$$

$$\simeq \phi_1(k)\, e^{ikR_1}\, \hat{a}_1 + \phi_2(k)\, e^{ikR_2}\, \hat{a}_2$$

$$\simeq \phi_0(k) \left[e^{ikR_1}\, \hat{a}_1 + e^{ikR_2}\, \hat{a}_2 \right]$$

Detection of <u>1st particle</u>:

$$n_1(r) = \langle \hat{\psi}^\dagger_{out}(r)\, \hat{\psi}_{out}(r) \rangle = |\phi_0(k)|^2 \cdot \left[e^{-ikR_1}\, \hat{a}_1^\dagger + e^{-ikR_2}\, \hat{a}_2^\dagger \right] \cdot$$

$$\cdot \left[e^{ikR_1}\, \hat{a}_1 + e^{ikR_2}\, \hat{a}_2 \right] =$$

$$= |\phi_0(k)|^2 \cdot \left[\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + e^{ik(R_1-R_2)}\, \hat{a}_2^\dagger \hat{a}_1 + \right.$$
$$\left. + e^{-ik(R_1-R_2)}\, \hat{a}_1^\dagger \hat{a}_2 \right]$$

$$\overset{\text{Fock}}{=} |\phi_0(k)|^2 \cdot (N_1 + N_2) \quad \text{without fringes.}$$

Detection of <u>2nd particle</u> at r' after a 1st at r:

a) Is distribution $n_2(r'|r)$ flat?

$$n_2(r'|r) = \langle \hat{\psi}^\dagger_{out}(r)\, \hat{\psi}^\dagger_{out}(r')\, \hat{\psi}_{out}(r')\, \hat{\psi}_{out}(r) \rangle =$$

$$= |\phi_0(k)|^2\, |\phi_0(k')|^2 \left\langle \left[e^{-ikR_1}\, a_1^\dagger + e^{-ikR_2}\, a_2^\dagger \right] \cdot \right.$$

$$\cdot \left[e^{-ik'R_1}\, a_1^\dagger + e^{-ik'R_2}\, a_2^\dagger \right] \left[e^{ik'R_1}\, \hat{a}_1 + e^{ik'R_2}\, \hat{a}_2 \right] \cdot$$

$$\left. \cdot \left[e^{ikR_1}\, \hat{a}_1 + e^{ikR_2}\, a_2 \right] \right\rangle =$$

right!

$$= |\phi_0(r)|^2 \, |\phi_0(r')|^2 \cdot \Big[N_1(N_1 - \cancel{1}) + N_2(N_2 - \cancel{1}) + N_1 N_2 \cdot$$

$$\Big(2 + e^{-ikR_1} e^{-ik'R_2} e^{ikR_2} e^{ik'R_1} +$$

$$+ e^{-ikR_2} e^{-ik'R_1} e^{ikR_1} e^{ik'R_2} \Big) \Big] =$$

$$= |\phi_0(r)|^2 \, |\phi_0(r')|^2 \Big[N_1^2 + N_2^2 + 2N_1N_2 \cdot$$

$$\cdot \Big(1 + \cos\big(k(R_2 - R_1) - k'(R_2 - R_1)\big) \Big) \Big] =$$

$$= |\phi_0(r)|^2 \, |\phi_0(r')|^2 \Big[N_1^2 + N_2^2 + 2N_1N_2 \big(1 + \cos\big((k-k')(R_2 - R_1)\big)\big) \Big]$$

fringes!

fringe period : $\Delta k = \frac{2\pi}{d}$, i.e. $\Delta z = \frac{2\pi\hbar t}{m d}$ as before

Fringes are deeper and deeper as one considers :

$n_M(r | z_1 \dots z_{M-1})$ for higher values of M → MC for large N

Ketterle's experiment → measures ALL particles

Fringes encoded in higher-order correlations. No trivial decomposition of n_M as product of n_1's

Density matrix decomposition on phase states :

$$\int \frac{d\theta}{2\pi} \, \Big| N : \frac{1}{\sqrt{2}}(1, e^{i\theta}) \Big\rangle \Big\langle N : \frac{1}{\sqrt{2}}(1, e^{i\theta}) \Big| =$$

$$= \int \frac{d\theta}{2\pi} \sum_{n,n'} \frac{1}{N!\,2^N} \binom{N}{n}\binom{N}{n'} (a_1^\dagger)^n (a_2^\dagger)^{N-n} |vac\rangle\langle vac| \, a_1^{n'} a_2^{N-n'} \cdot e^{i\theta(N-n)} \cdot e^{-i\theta(N-n')} =$$

$$= \sum_n \frac{1}{2^N} \binom{N}{n}^{\cancel{2}} \frac{n!\,(N-n)!}{\cancel{N!}} \, |n, N-n\rangle\langle n, N-n| =$$

$$= \sum_n \frac{1}{2^N} \frac{N!}{n!(N-n)!} \quad |n, N-n\rangle \langle n, N-n|$$

for high enough N $\longrightarrow$ peaked around $N/2$,

width $\sim \frac{\sqrt{N}}{2}$

$\longrightarrow$ same physics as with Fock at $N/2$

$$n_1(r|\theta) = |\phi_0(k)|^2 \left[\frac{N}{2} + \frac{N}{2} + \left(\frac{N}{2} e^{ik(R_1-R_2)} e^{-i\theta} + h.c. \right) \right]$$

$$= N|\phi_0(k)|^2 \left[1 + \cos\left(k(R_1-R_2)-\theta\right) \right]$$

$$\langle n_1(r|\theta)\rangle_\theta = N|\phi_0(k)|^2$$

$$n_2(r'|r|\theta) = N^2 |\phi_0(k)|^2 |\phi_0(k')|^2 \cdot \left| e^{-ikR_1} + e^{-ikR_2} e^{i\theta} \right|^2 \cdot$$

$$\cdot \left| e^{-ik'R_1} + e^{-ik'R_2} e^{i\theta} \right|^2$$

which now factorizes into $n_1(r|\theta)$'s.

$$\langle n_2(r'|r|\theta)\rangle_\theta = N^2 |\phi_0(k)|^2 |\phi_0(k')|^2 \cdot \Big\langle \left(2 + \left(e^{ik(R_2-R_1)} e^{-i\theta} + h.c.\right)\right) \cdot$$

$$\cdot \left(2 + \left(e^{ik'(R_2-R_1)} e^{-i\theta} + h.c.\right)\right) \Big\rangle_\theta =$$

$$= N^2 |\phi_0(k)|^2 |\phi_0(k')|^2 \cdot \left[4 + \left(e^{i(k-k')(R_2-R_1)} + h.c.\right) \right] =$$

$$= N^2 |\phi_0(k)|^2 |\phi_0(k')|^2 \cdot 2 \left[2 + \cos(k-k')(R_2-R_1) \right]$$

in agreement with Fock State prediction.

In conclusion:

Spontaneous Symmetry Breaking point of view is convenient for calculations and gives accurate predictions

- factorised expressions
- average over θ to be performed

even if phase states are NOT eigenstates of H!

Phase diffusion:

$$|\Psi_m\rangle = \frac{1}{\sqrt{N!\,2^N}} \left(\hat{a}_1^\dagger + e^{i\theta}\hat{a}_2^\dagger\right)^N |vac\rangle =$$

$$= \frac{1}{2^{N/2}} \frac{1}{\sqrt{N!}} \sum_{n=0}^{N} \frac{N!}{n!\,(N-n)!} \left(a_1^\dagger\right)^{N-n} e^{in\theta} \left(a_2^\dagger\right)^n |vac\rangle$$

$$= \frac{1}{2^{N/2}} \sum_{n=0}^{N} \sqrt{\frac{N!}{n!\,(N-n)!}}\; e^{in\theta}\; |1:N-n,\ 2:n\rangle$$

$$E(n) = E(N/2) + \mu\,(n - N/2) + \frac{1}{2}\frac{\partial\mu}{\partial N}\,(n-N/2)^2 + \dots$$

$$E_1(N-n) + E_2(n) = E_1(N/2) + E_2(N/2) + \mu_1(N-n-N/2) + \mu_2(n-N/2) +$$

$$+ \frac{1}{2}\frac{\partial\mu_1}{\partial N}(N-n-N/2)^2 + \frac{1}{2}\frac{\partial\mu_2}{\partial N}(n-N/2)^2 + \dots$$

$$\simeq 2E(N/2) + \frac{\partial\mu}{\partial N}(n-N/2)^2 + \dots$$

$$|\Psi(t)\rangle = \frac{1}{2^{N/2}} \sum_{n=0}^{\infty} \sqrt{\frac{N!}{n!\,(N-n)!}}\; e^{in\theta}\, e^{-i\frac{\partial\mu}{\partial N}(n-N/2)^2 t}\, e^{-2iE(N/2)t}\;\cdot$$

$$\cdot\, |1:N-n,\ 2:n\rangle$$

Analogy with massive particle:

$$\theta \longrightarrow x$$

$$n \longrightarrow p \qquad \text{(or, better } n - N/2 \longrightarrow p\text{)}$$

$$H_{eff} = \frac{1}{2M^*} p^2 \qquad \text{with} \qquad M^* = \frac{1}{2}\left(\frac{\partial \mu}{\partial N}\right)^{-1}$$

$\hookrightarrow$ spreading of θ in time t_{coll} to $\simeq 1 \longrightarrow$ lose phase memory

$$\Delta\theta^2(t) = \Delta\theta^2(0) + \frac{\Delta n^2}{M^{*2}} \cdot t^2 = \Delta\theta^2(0) + 2N\left(\frac{\partial \mu}{\partial N}\right)^2 \cdot t^2$$

$$t_{coll} = \left(\frac{1}{2N\left(\frac{\partial\mu}{\partial N}\right)^2}\right)^{1/2} = \frac{1}{\sqrt{2N}} \cdot \frac{1}{\partial\mu/\partial N}$$

Uniform BEC: $\mu = \frac{N}{V} g$, $\frac{\partial\mu}{\partial N} = \frac{g}{V}$

$$\Longrightarrow t_{coll} = \frac{1}{\sqrt{2N}} \cdot \frac{V}{g} = \sqrt{\frac{V}{2ng^2}}$$

which $\to \infty$ in T.D. limit $V \to \infty$ at fixed n, g.

NOTE: quantum nature appears in revivals at $t_{rev} = 1/(\partial\mu/\partial N) \propto \frac{V}{g}$ which grows faster than t_{coll} in T.D. limit.

NOTE 2: at $t = t_{rev}/2$ system in Schrödinger cat state

$$|\psi(t_{rev}/2)\rangle \sim |\theta\rangle + |\theta+\pi\rangle$$

Bibliography

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