Lecture 8 : Excursions in a strongly interacting world

1D system $\longrightarrow$ allows for simple analytical understanding

$\longrightarrow$ simple form of 2-body scattering wf

$$V(x) = V_0\,\delta(x)$$

$$-\frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial x^2}\psi(x) + V_0\,\delta(x)\,\psi(x) = E\,\psi(x)$$

$$m^* = m/2 = \text{reduced mass}$$

$$\int_{0^-}^{0^+}\left(-\frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial x^2}\psi(x) + V_0\,\delta(x)\,\psi(x)\right)dx = \int_{0^-}^{0^+}\left(E\,\psi(x)\right)dx$$

$$-\frac{\hbar^2}{2m^*}\left(\frac{d}{dx}\psi(x)\Big|_{0^+} - \frac{d}{dx}\psi(x)\Big|_{0^-}\right) + V_0\,\psi(0) = 0$$

$$\psi'(0+) - \psi'(0-) = \frac{2m\,V_0}{\hbar^2}\psi(0)$$

look for bosonic, symmetric wavefunction : $\psi(x) = \psi(-x)$

$$\left.\begin{aligned} \psi(x) &= e^{ikx} + r\,e^{-ikx} & (x<0) \\ \psi(x) &= e^{-ikx} + r\,e^{ikx} & (x>0) \end{aligned}\right\}\quad \frac{\hbar^2 k^2}{2m^*} = E$$

$$\psi'(0^+) - \psi'(0^-) = (ik - ikr) - (-ik + ikr) = 2ik(1-r)$$

$$\psi(0) = 1+r$$

$$\Rightarrow\ 2ik(1-r) = \frac{2m\,V_0}{\hbar^2}(1+r)$$

$$\cancel{2}ik - \frac{\cancel{2}mV_0}{\hbar^2} = \left(\frac{\cancel{2}mV_0}{\hbar^2} + \cancel{2}ik\right) \cdot r$$

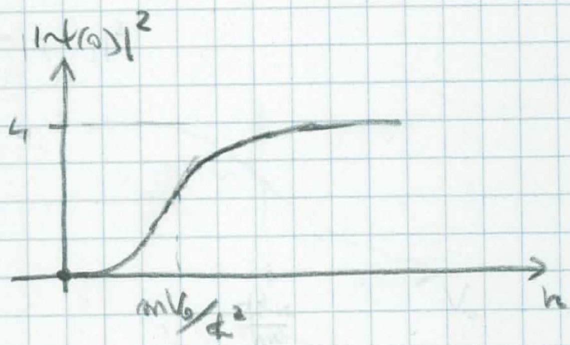
$$r = \frac{ik - mV_0/\hbar^2}{ik + mV_0/\hbar^2}$$

As expected by particle number conservation, $|r|^2 = 1$

$$\psi(0) = 1 + r = 1 + \frac{ik - mV_0/\hbar^2}{ik + mV_0/\hbar^2} = \frac{ik + \cancel{mV_0/\hbar^2} + ik - \cancel{mV_0/\hbar^2}}{ik + mV_0/\hbar^2}$$

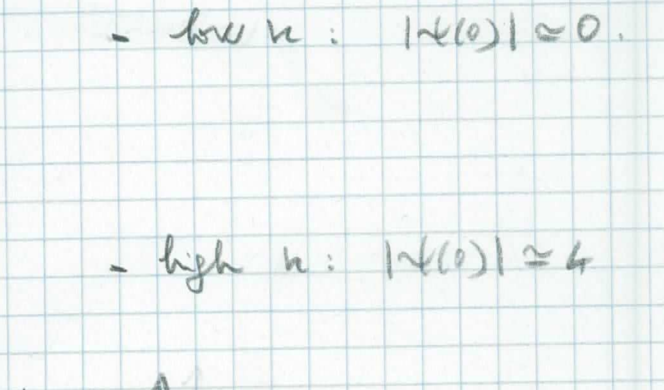
$$= \frac{2ik}{ik + mV_0/\hbar^2}$$

$$|\psi(0)|^2 = \frac{4k^2}{k^2 + \left(mV_0/\hbar^2\right)^2}$$



- low k : $|\psi(0)| \simeq 0$. Potential effective in repelling 2 particles.

- high k : $|\psi(0)| \simeq 4$ as for non-interacting particles.



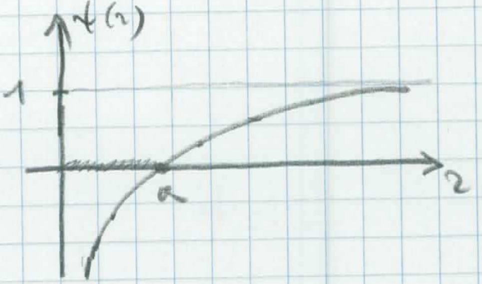
Phase shift goes to $-\pi$ at low momenta.

$\Rightarrow$ for any value of V_0 low-energy interactions are strong!

$\Rightarrow$ for large value of V_0, wave function regular, tends to $\psi(x) = 2i\,|\sin(kx)|$

This is a peculiarity of <u>1D</u> quantum physics.

<u>In 3D</u>:



$\psi(r) \simeq 1 - a/r$

⇒ scattering cross section $\sigma = 8\pi a^2$ at low energy tends to 0 in non-interacting gas limit

i.e. $\lim_{k \to 0}$ and $\lim_{a \to 0}$ commute

(not in 1D ...)

<u>Strong interaction limit $V_0 \to \infty$</u>:

$\psi(x) = |\sin kx|$ is zero at $x = 0$

goes into fermionic wf if $|\ \ |$ is eliminated

⟶ Simplest example of <u>Girardeau's theorem</u>.

Bosonic many-body wf $\psi(x_1 .. x_N)$ for $V_0 \to \infty$ is related to Fermionic wf without interactions by $\psi(x_1 .. x_N) = \varepsilon(\sigma)\ \psi_F(x_1 .. x_N)$

where σ = permutation to sort $x_1 .. x_N$ into ascending order.

One-to-one mapping:

eigenstate of bosonic problem at $V_0 = \infty$

$\updownarrow$

eigenstate of non-interacting fermion system

} with the same energy

Homogeneous system:

$$\psi_F(x_1 \ldots x_N) = \mathcal{A}\left[e^{iq_1x_1} \ldots e^{iq_Nx_N}\right]$$

$q_1 \ldots q_N$ determined by boundary conditions.

Ground state : $q_1 \ldots q_N$ form a Fermi sphere

Density $\longrightarrow$ homogeneous

Density - density correlations $\longrightarrow$ Pauli hole

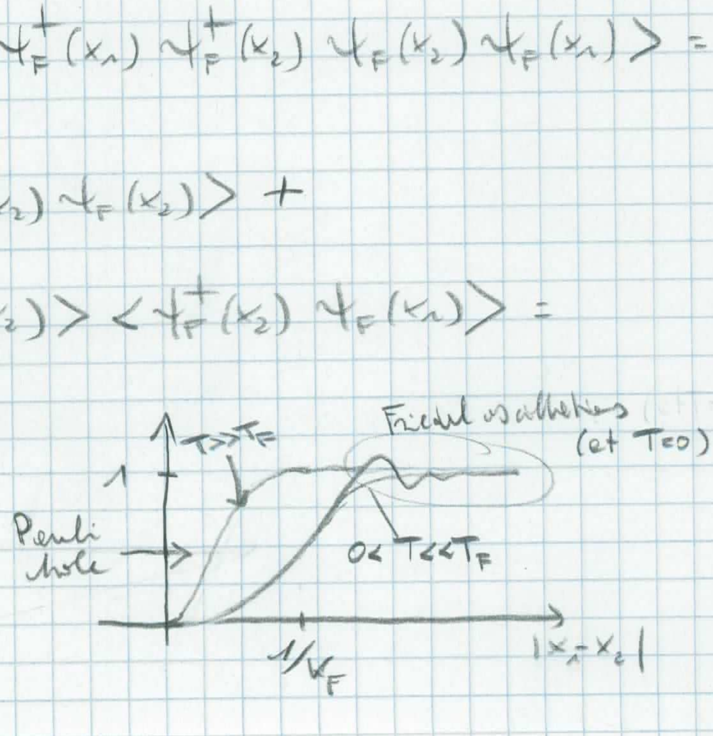
} spatial properties preserved in F-B mapping

$$\langle \psi^\dagger(x_1)\psi^\dagger(x_2)\psi(x_2)\psi(x_1)\rangle = \langle \psi_F^\dagger(x_1)\psi_F^\dagger(x_2)\psi_F(x_2)\psi_F(x_1)\rangle =$$

$$\overset{\text{Wick}}{=} \langle \psi_F^\dagger(x_1)\psi_F(x_1)\rangle\langle \psi_F^\dagger(x_2)\psi_F(x_2)\rangle +$$

$$- \langle \psi_F^\dagger(x_1)\psi_F(x_2)\rangle\langle \psi_F^\dagger(x_2)\psi_F(x_1)\rangle =$$

$$= n^2\left(1 - |g^{(1)}(x_1-x_2)|^2\right)$$



Analytic forms

$$\begin{cases} g^{(1)}(x) = \dfrac{\sin k_F x}{\pi x} & (T=0) \\[2ex] g^{(1)}(x) = \dfrac{k_F T}{2T_F} \dfrac{\sin k_F x}{\sinh\left(\dfrac{\pi T}{2T_F} k_F x\right)} & 0<T\ll T_F \\[2ex] g^{(1)}(x) = \exp\left(-\pi x^2/\lambda_T^2\right) & T_F \ll T_F \end{cases}$$

* for $T \ll T_F$ $\longrightarrow$ cut off of Friedel oscillations

at $x \simeq \ell_c = 2T_F/\pi T k_F$

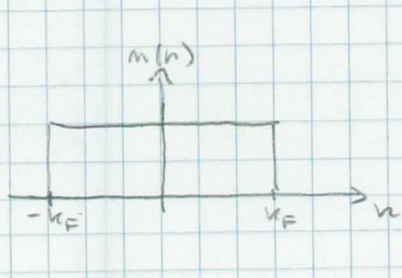
* long-distance decay $\begin{cases} 1/x & \text{for } T=0 \\ \exp(-x/\ell_c) & \text{for } 0<T\ll T_F \\ \exp(-\pi^2 x^2/\lambda_T^2) & \text{for } T_F \ll T. \end{cases}$

↳ Negative Hanbury-Brown and Twiss effect

Signature of "fermionization" is reduction of 3-body losses: particles are never close, so they can not decay into molecules.
(Pauli hole even more visible in $g^{(3)}$)

Momentum distribution:

Fermi model



but $\rho^{(1)}(x,x') = \int dx_2 \dots dx_N \, \psi(x, x_2 \dots x_N)\, \psi^*(x', x_2 \dots x_N)$

is hardly written in terms of $\rho_F^{(1)}(x,x')$.

Girardeau's theorem is straightforward on diagonal elements of $\rho^{(1)}(x;x)$, $\rho^{(2)}(x,x';x,x')$ ..., but involves non-trivial $\pm$ for non-diagonal elements

Complicated calculations $(T=0)$:

$$n(k) = \begin{cases} \sim 1/k^4 & \text{for } k \to \infty \\ 1/\sqrt{k} & \text{for } k \to 0 \end{cases}$$

$\rightarrow$ large-k behaviour dominated by 2-body scattering ...: δ-peak in $\psi''(x_1-x_2)$

$\hookrightarrow \psi(k) \sim 1/k^2 \longrightarrow |\psi(k)|^2 \sim 1/k^4$

$\rightarrow$ low-k behaviour to be compared with:

* Thermal, non-interacting gas $n(k) \sim \frac{k_B T}{\hbar^2 k^2/2m}$

* $T=0$, interacting gas: $n(k) \sim \sqrt{\frac{2mc_s}{\hbar k}}$

* $T>0$, interacting gas: $n(k) \sim \frac{k_B T}{\hbar^2 k^2/2m}$

$\Rightarrow$ Tonks-Girardeau gas has slower behaviour for $k \to 0$, which reminds fermionic analogy.

In real space : $\rho(x,x') \sim \frac{An}{(k_F|x-x'|)^{1/2}} \xrightarrow{|x-x'| \to \infty} 0$: no BEC, but strong finite size effects.

at finite $T>0$, exponential decay $\rho(x,x') \sim \exp(-|x-x'|/\ell_c)$.

How to estimate value of V_0 needed to enter T-G regime?

- 2 energy scales : $\begin{cases} \varepsilon_F = \hbar^2 k_F^2 / 2m = \frac{\hbar^2}{2m} \pi^2 n^2 \\ \varepsilon_{int} = n V_0 \end{cases}$

$$\Rightarrow \begin{cases} \text{- if } \varepsilon_F \gg \varepsilon_{int} \longrightarrow \text{dilute, weakly-interacting gas.} \\ \text{- if } \varepsilon_F \ll \varepsilon_{int} \longrightarrow \text{strongly interacting gas.} \end{cases}$$

$$\gamma = \varepsilon_{int} / \varepsilon_F = \frac{n V_0}{\hbar^2 \pi^2 n^2 / 2m} = \frac{2m V_0}{\hbar^2 \pi^2 n} \sim n^{-1}$$

$\hookrightarrow$ lower density, stronger interacting!!

* different from 3D : dilution parameter $(n a^3)^{1/2} \sim n^{1/2}$

* physical interpretation : typical relative momentum $\sim k_F$, max momentum at which $|r| \sim 1$ is $m V_0 / \hbar^2 = k_{max}$

$$\Rightarrow k_{max} / k_F = \frac{m V_0}{\hbar^2} \frac{1}{\pi n} = \pi \gamma$$

$\hookrightarrow$ low density : particle rebound onto each other

$\hookrightarrow$ high density : particles can cross each other

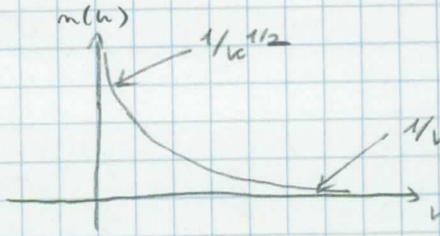
$\hookrightarrow$ phenomenology related with non-commutation of limits

$$\lim_{V_0 \to 0} \lim_{k \to 0} \quad \text{vs.} \quad \lim_{k \to 0} \lim_{V_0 \to 0} \quad r(k, V_0).$$

low density — weak interactions

(Some) experimental observations

1) Momentum distribution



* "ideal" world :

* feature at $k \to \infty$ not characteristic of T-G

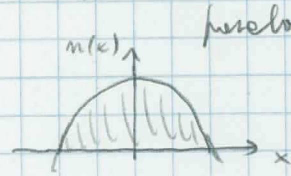
* feature at $k \to 0$ easily masked by finite-size effects

$\Longrightarrow$ complicate fitting procedure necessary!

see e.g. B. Paredes et al, Nature 429, 277 (2004)

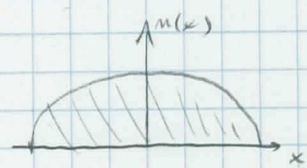
2) System size in harmonic trap

* BEC : $n(x) \simeq \dfrac{\mu - \frac{1}{2} m \omega_0^2 x^2}{g} = A - Bx^2$ inverted parabola



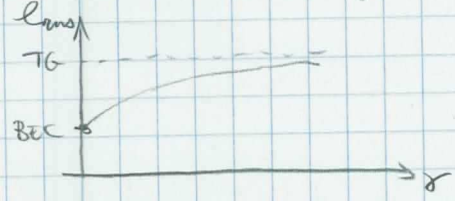
* T-G : $\mu = \varepsilon_F = \dfrac{\hbar^2 \pi^2}{2m} n^2$

$$\Longrightarrow \quad n(x) = \sqrt{A' - B' x^2}$$



$\longrightarrow$ difficult to distinguish the two when averaging over many 1D tubes is performed : in both cases distribution $\simeq$ gaussian.

→ better to measure rms length of system.



see, e.g. T. Kinoshita, T. Wenger, D. S. Weiss, Science 305, 1125 (2004)

3) Average kinetic energy after expansion :

* equal to total initial energy (long-time interaction energy vanishes).

→ slightly > 0 because of trap 0-point energy

see, e.g. Kinoshita et al op. cit

4) Other strategies : check if experiment exist !!!

* speed of sound : $\underset{BEC}{c_s = \sqrt{\frac{gn}{m}}} \longrightarrow \underset{TG}{\frac{\hbar k_F}{m}}$

* 3-body recombination : $\Gamma^{(3)} n^3 \longrightarrow \Gamma^{(3)} n^3 \cdot g^{(3)}(0) \simeq 0$.

Bibliography

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- Y. Castin, "Simple theoretical tools for low-d Bose gases", lecture notes of 2003 Les Houches school, cond-mat/0407118.

- Experiments :

* B. Paredes et. al. Nature <u>429</u>, 277 (2004)
* T. Kinoshita et al. Science <u>305</u>, 1125 (2004)
* B. Laburthe - Tolra et al , Phys. Rev Lett <u>92</u>, 190401 (2004)