Bogoliubov theory of small fluctuations:

→ quantized Hamiltonian $H = E_0 + \sum_n \hbar \varepsilon_n^{Bog} b_n^+ b_n$

not possible because of pump / losses

→ more complex formalism of master equation
Not completely developed yet.

→ mean field equations + linearization around stationary solution

Homogeneous case

$$i \frac{\partial}{\partial t} \psi = -\frac{\hbar \nabla^2}{2m} \psi + g |\psi|^2 \psi + i \left(P_0 - \gamma - P_1 |\psi|^2 \right) \psi + \omega_0 \psi$$

below threshold : equilibrium $\psi = 0$

$$i \frac{\partial}{\partial t} \delta\psi = -\frac{\hbar \nabla^2}{2m} \delta\psi + \emptyset + i (P_0 - \gamma) \delta\psi + \omega_0 \psi$$

$$\omega \, \delta\psi_k = \frac{\hbar k^2}{2m} \delta\psi_k + i (P_0 - \gamma) \delta\psi_k + \omega_0 \delta\psi_k$$

$$\omega = \omega_0 + \frac{\hbar k^2}{2m} + i (P_0 - \gamma)$$

free-particle dispersion

loss rate : tends to 0 as $P_0 \to \gamma^-$
critical slowing down

above threshold : $|\bar{\psi}|^2 = \frac{P_0 - \gamma}{P_1}$

$$\bar{\omega} = \omega_0 + g|\bar{\psi}|^2$$

$$\psi(x,t) = e^{-i\bar{\omega}t}\left[\bar{\psi} + \delta\psi(x,t)\right]$$

$$i\frac{\partial}{\partial t}\delta\psi + \cancel{\bar{\omega}\delta\psi} = -\frac{\hbar\nabla^2}{2m}\delta\psi + \cancel{g|\bar{\psi}|^2\delta\psi} + i\,(\cancel{P_0 - \gamma - P_1|\bar{\psi}|^2})\delta\psi + \cancel{\omega_0\delta\psi} +$$

$$+ g|\bar{\psi}|^2\delta\psi + g\bar{\psi}^2\delta\bar{\psi} - iP_1|\bar{\psi}|^2\delta\psi - iP_1\bar{\psi}^2\delta\psi^*$$

$$i\frac{\partial}{\partial t}\delta\psi = -\frac{\hbar\nabla^2}{2m}\delta\psi + (g - iP_1)\left(|\bar{\psi}|^2\delta\psi + \bar{\psi}^2\delta\psi^*\right)$$

assume for simplicity $\bar{\psi} \in \mathbb{R}$, $\bar{n} = |\bar{\psi}|^2$

$$i\frac{\partial}{\partial t}\begin{pmatrix}\delta\psi \\ \delta\psi^*\end{pmatrix} = \begin{pmatrix} -\frac{\hbar\nabla^2}{2m} + (g - iP_1)\bar{n} & , & (g - iP_1)\bar{n} \\ (-g - iP_1)\bar{n} & , & +\frac{\hbar\nabla^2}{2m} + (-g - iP_1)\bar{n}\end{pmatrix}\begin{pmatrix}\delta\psi \\ \delta\psi^*\end{pmatrix}$$

Plane-wave solution at k: $-\frac{\hbar\nabla^2}{2m} \longrightarrow \frac{\hbar k^2}{2m}$

$$\left(\frac{k^2}{2m} + (g - iP_1)\bar{n} - \lambda\right)\left(-\frac{k^2}{2m} - (g + iP_1)\bar{n} - \lambda\right) +$$

$$- (g - iP_1)\bar{n}\,(-g - iP_1)\bar{n} = 0$$

$$\left(\lambda - \left(\frac{k^2}{2m} + g\bar{n}\right) + iP_1\bar{n}\right)\left(\lambda + \left(\frac{k^2}{2m} + g\bar{n}\right) + iP_1\bar{n}\right) + (g\bar{n})^2 + P_1^2\bar{n}^2 = 0$$

$$\left((\lambda + iP_1\bar{n}) - \left(\frac{k^2}{2m} + g\bar{n}\right)\right)\left(\lambda + iP_1\bar{n}) + \left(\frac{k^2}{2m} + g\bar{n}\right)\right) + (g\bar{n})^2 + P_1^2\bar{n}^2 = 0$$

$$(\lambda + i P_1 \bar{n})^2 - \left(\frac{k^2}{2m} + g\bar{n}\right)^2 + g^2\bar{n}^2 + P_1^2 \bar{n}^2$$

$$(\lambda + i P_1 \bar{n})^2 = \varepsilon_{Bog}(k)^2 - P_1^2 \bar{n}^2$$

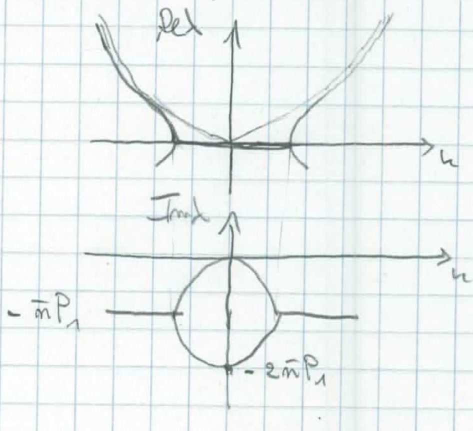
$$\boxed{\lambda = -i P_1 \bar{n} \pm \sqrt{\varepsilon_{Bog}(k)^2 - P_1^2 \bar{n}^2}}$$

* satisfies Goldstone theorem $\lambda(k \to 0) = -i P_1 \bar{n} \pm \sqrt{-P_1^2 \bar{n}^2}$
  $= 0, -2i P_1 \bar{n}$

* dynamical stability : $\exp(-i\lambda t) \simeq \exp(-i(-i P_1 \bar{n})t)$
  $= \exp(-P_1 \bar{n} t)$

* $P_1 \bar{n} = P_0 - \gamma$. Tends to 0 as $P_0 \to \gamma^+$
  critical slowing down.

* Diffusive dispersion for low k such that $\varepsilon_{Bog}(k) < P_1 \bar{n}$.



[Wouters and IC, PRL 99 140402 (07)]

* Superfluid or not ?
  - Landau $\longrightarrow$ $v_c = \min_k \frac{\mathrm{Re}[\omega(k)]}{k} = 0$
  - numerically $\longrightarrow$ threshold-like behaviour of drag force $F(v)$
    $\longrightarrow$ crucial role of $\mathrm{Im}[\omega(k)]$ [Wouters, IC. arXiv: 0707.1446]

Eigenvectors:

$$\left(\frac{\hbar k^2}{2m} + (g - iP_1)\bar{n} - \lambda\right) U_k + (g - iP_1)\bar{n} V_k = 0$$

$$\left(\frac{\hbar k^2}{2m} + (g - iP_1)\bar{n} + iP_1\bar{n} \mp \sqrt{\varepsilon_{Bog}(k)^2 - P_1^2\bar{n}^2}\right) U_k + (g - iP_1)\bar{n} V_k = 0$$

$$\left(\frac{\hbar k^2}{2m} + g\bar{n} \mp \sqrt{\varepsilon_{Bog}(k)^2 - P_1^2\bar{n}^2}\right) U_k + (g - iP_1)\bar{n} V_k = 0$$

i) $\varepsilon_{Bog}(k) \ll P_1\bar{n}$

$$(g\bar{n} \mp iP_1\bar{n})\, U_k + (g - iP_1)\bar{n} V_k = 0$$

(−) Goldstone mode : $(g - iP_1)\bar{n}\, U_k + (g - iP_1)\bar{n} V_k = 0$

$$\Longrightarrow \quad U_k + V_k = 0$$

$$\delta\psi(x,t) = U_k e^{ikx} e^{-i\omega t} b_k + V_k^* e^{-ikx} e^{i\omega t} b_k^\dagger$$

$$\delta\rho = \bar{\psi}^* \delta\psi + \bar{\psi}\,\delta\psi^* = \bar{\psi}\,(U_k + V_k)\left(e^{ikx} e^{-i\omega t} b_k + h.c.\right)$$

$\bar{\psi}^* \to \in \mathbb{R}$

$$= 0$$

→ no density modulation

→ Goldstone mode corresponds to phase rotation.

(+) mode, damped $-2iP_1\bar{n}$

$$(g + iP_1)\cancel{\bar{n}}\, U_k + (g - iP_1)\cancel{\bar{n}}\, V_k = 0$$

$$g(U_k + V_k) + iP_1(U_k - V_k) = 0$$

$$U_k + V_k = -\frac{iP_1}{g}\frac{1}{U_k + V_k}$$

$$u_k + v_k = \sqrt{-i\frac{P_1}{g}} \quad , \quad u_k - v_k = \sqrt{\frac{ig}{P_1}}$$

* for $g \to 0$: $u_k - v_k \to 0$ purely density mode
* finite g : interactions couple density and phase

<u>Open question</u> : role of fluctuations in destabilizing BEC

<u>equilibrium</u> : $H_{Bg} = \sum_k \omega_{Bg}(k)\, b_k^+ b_k$

→ ground and thermal states determined by $\exp(-\beta H_{Bg})$

→ non-trivial zero-point physics from Bogoliubov transformation $\delta\psi = \ldots b_k + \ldots b_k^+$

→ quantum (and thermal) depletion of BEC

<u>non-equilibrium</u> : dynamics matters in determining excitation of Bogoliubov modes

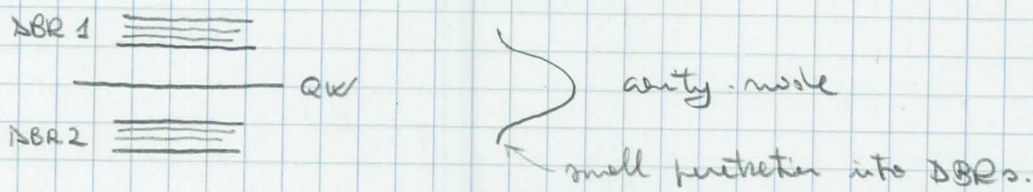
→ requires solution of master equation

→ Bogoliubov approach can be recast in language of stochastic differential eqs for quasi-probability distribution, i.e. Wigner-W. Approximations needed to bring it into positive-2-order-diffusion form.

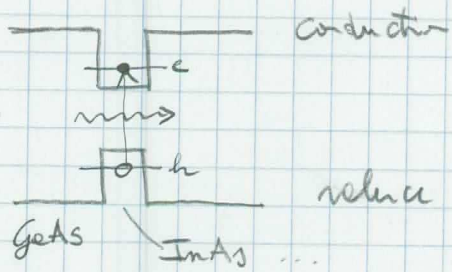
Exciton - polaritons

cavity with embedded quantum wells.

DBR mirrors (finite Photonic - band-gap crystals)
↳ cavity = defect



Quantum well



- e-h pair bound by Coulomb into exciton.
- delocalized over quantum well, momentum k defined by photon.
- nm-sized Bohr radius (ε_∞ material, $m_e^*/m_e \ll 1$)

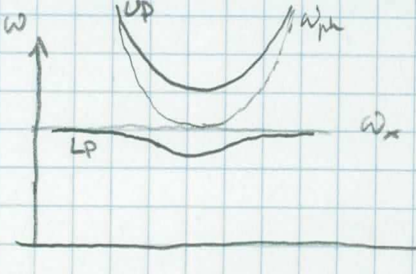
$$|\psi_{exc}\rangle = \sum_q \tilde{\Phi}(q) \cdot \psi_e^\dagger(k/2+q)\, \psi_h^\dagger(k/2-q)\ |\text{ground}\rangle$$

$$\tilde{\Phi}(q) = \text{F.T. } \Phi(r) \quad \text{hydrogen-like}$$

- sharp line located inside semiconductor band-gap, binding energy $\simeq$ a few meV's

- weak dispersion $\varepsilon_x(k)$ as compared to photon one, $m_x \simeq m_e^* \gg m_{ph}^*$

- momentum conservation along cavity plane :
  photon state at k $\longleftrightarrow$ exciton state at k,
  no spontaneous emission.



→ mixed states are called

EXCITON - POLARITONS.

(UP = upper polariton, LP = lower polariton)

- photon component → allows for optical excitation and diagnostics from emitted light.

$$\hat{E}_{emitted}(k) \simeq \alpha \hat{E}_{cav}(k) \simeq$$
$$\alpha \left(U_{ph}^{LP}(k)\, \hat{\psi}_{LP}(k) + U_{ph}^{UP}(k)\, \hat{\psi}_{UP}(k) \right)$$

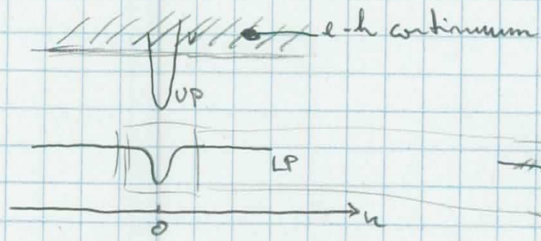
- exciton component provides optical nonlinearity + amplification mechan.

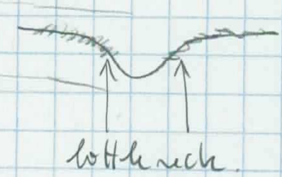
→ excitons ≃ particles of size a_B, collide elastically under Coulomb + Pauli filling.

→ saturation of transition at high density.

↳ experimentally: interactions appear repulsive

Excitation process





* excite e-h pairs by laser / strong thermal light
* relax into LP. Because of kinetics, accumulate at "bottleneck" points

↳ reservoir of amplification

* collisions: 2 bottleneck polaritons → 1 polariton at bottom of LP + 1 energetic exciton

* weakly energy dependent scattering process. Lowest states favoured

→ can condense around $k = 0$ for high pump rate

* for high enough density → observe a Boltzmann-Bose distribution of LP's at bottom of band.

↳ initially believed signature of thermalization by LP-LP collisions

↳ later experiments showed same phenomenology in a VCSEL of non-interacting photons.
[D. Bajoni et al. PRB 75, 201305 ('07)]

↳ originates from condensation kinetics, possibly from energy-dependence of amplification.

Many OPEN QUESTIONS...

POLARITON LASING

EXCITON-(POLARITON) BEC

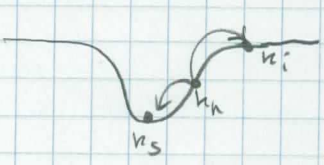
LASING

Many confusing statements in the literature...

Polariton OPO:

pump at magic angle k_p allows for simultaneous:

$$\left.\begin{array}{ll} \text{energy} & \mathcal{E}(k_s) + \mathcal{E}(k_i) = 2\,\mathcal{E}(k_p) \\ \text{and momentum} & k_s + k_i = 2k_p \end{array}\right\} \text{conservation}$$



$$\begin{cases} k_s = \text{signal} \\ k_i = \text{idler} \end{cases}$$

Parametric Hamiltonian

$$H_{par} = g\, a_s^\dagger a_i^\dagger a_p^2 + h.c.$$

creates a pair of s/i polaritons

Hamiltonian symmetric under $a_{s,i} \longrightarrow a_{s,i}\, e^{\pm i\phi}$; $U(1)$

but symmetry spontaneously broken above the parametric threshold when coherent signal / idler light start being emitted.

[IC and C. Ciuti, PRB 72, 125335 ('05)]

Spatially extended OPO :

$$H_{par} = g \int d^2r \; \psi_s^\dagger(r)\, \psi_i^\dagger(r)\, \psi_p(r)\, \psi_p(r) + h.c.$$

again : $\psi_{s,i}$ order parameter, BEC signalled by $\langle \psi_s^\dagger(r)\, \psi_s(r') \rangle$

Threshold is critical point

Bogoliubov dispersion, diffusive Goldstone mode
[M. Wouters and IC, PRA 76, 043807 ('07)]
quasi-BEC effects in lower-d
[Wouters and IC, PRB 74, 245316 ('06)]

NOTE degenerate OPO is different $H = g\, a_s^{\dagger 2} a_p^2$

only $\mathbb{Z}_2$ symmetry spontaneously broken (discrete)

↳ Ising oscillations are a d-OPO.

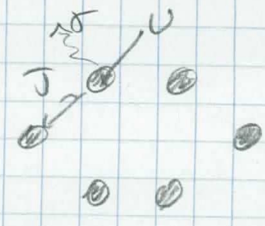
* First examples of polariton BEC were of the OPO-type

↘ $g^{(1)}$, $g^{(2)}$ measured at different points and times
[Baas et al. PRL 96, 176401 ('06) ; Krizhanovskii et al PRL 97, 097402 ('06)]

↘ pump and probe expts to measure Bogoliubov spectrum
[Ballarini et al, arXiv:0807.3224]

↘ propagation of Bogoliubov pulses: intriguing wavepacket dynamics
[Amo et al, arXiv:0711.1539 to appear in Nature]

Strongly interacting polariton gases



- necklace of M coupled cavities

losses γ

tunnel coupling J

on-site interactions U

coherent driving $F_p(x,t)$

$$H = \sum_{\langle i,j \rangle} -J a_i^\dagger a_j + U a_i^\dagger a_i^\dagger a_i a_i + \left(F_p(x,t)\, a_i^\dagger + h.c.\right)$$

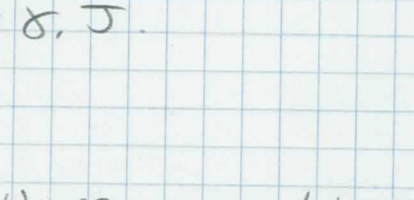
- theoretical predictions : small cavity size

$$\Longrightarrow \quad U \gg \gamma, J.$$

- impenetrable boson limit $U = \infty$, many-body states can be classified by momenta of fermions in Fermi-Bose mapping.

* mesoscopic case $\rightarrow \Delta E \gg \gamma$.

* spectrally resolved in Photoluminescence Excitation (PLE) experiment.

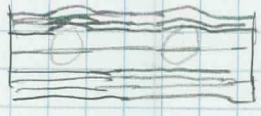


* single many-body state can be created by spectral selection : $\omega_p = E/N$ (N = # photon in state)

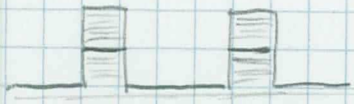
* properties inferred from emitted light (intensity correlations)

How to create system.

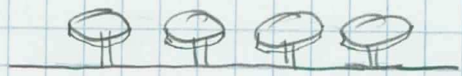
- laterally patterned cavities



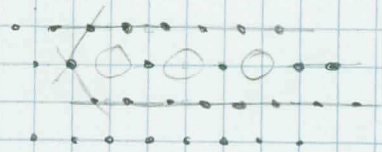
- micropillars



- arrays of microdisks



- photonic crystal cavities



... but also in μ-wave domain using superconducting waveguides + Cooper pair boxes.

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