Open (quantum) systems

27/7/26

$$\boxed{S} \longleftrightarrow \boxed{\;R\;} \qquad H = H_S + H_R + H_{SR}$$

weak

large / infinite reservoir
- continuous spectrum
- irreversible dynamics

example : friction forces

Direct solution $\psi \in \mathcal{H}_S \otimes \mathcal{H}_R$, Schröd eq.

difficult, extra non-interesting info

↳ eliminate reservoir → dynamics of system only

$$\rho_S = \mathrm{Tr}_R(\rho)$$

$$\frac{d}{dt}\rho_S = \frac{1}{i\hbar}[\rho_S, H_S] + \mathcal{L}(\rho_S)$$

__NOTE :__ ρ_S is NOT pure state due to entanglement with R

$\mathrm{Tr}(\rho_S) = 1$ → not non-Hermitian H.

Breuer - Petruccione "Theory of Open Quantum Systems"
CCT et al. "Atom-Photon interaction processes"
lecture notes on Quantum Optics, on my website.

Weisskopf-Wigner model of decay

$$H = \hbar\omega_e(\hat{\sigma}_z + 1/2) + \hbar g\, \hat{E}(0)\, \hat{\sigma}_x + \int \frac{dk}{2\pi}\, \hbar\omega_k\, \hat{a}_k^\dagger \hat{a}_k$$

$$\hat{E}(x) = \int \frac{dk}{2\pi}\left(e^{ikx}\hat{a}_k + e^{-ikx}\hat{a}_k^\dagger\right)$$ quantum field

$\hat{\sigma} \longrightarrow$ 2 level atom @ ω_e

ω_k dispersion of field.

RWA: $\hat{E}(0)\,\hat{\sigma}_x \propto \int \frac{dk}{2\pi}\, \hat{a}_k^\dagger \sigma^- + h.c.$

Start: $|\psi(t=0)\rangle = |e\rangle \otimes |vac\rangle$

What is $|\psi(t)\rangle$ @ generic t ?

Conservation of $\hat{N} = \hat{\sigma}_z + \int \frac{dk}{2\pi} a_k^\dagger a_k$

$$|\psi(t)\rangle = c_e |e; vac\rangle + \int \frac{dk}{2\pi}\, c_k \underbrace{|g; 1{:}k\rangle}_{|g\rangle \otimes a_k^\dagger |vac\rangle}$$

$$\int \frac{dk}{2\pi}\dot{c}_k |g, 1{:}k\rangle + i\hbar\dot{c}_e|e; vac\rangle = i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle = \hbar\omega_e c_e |e; vac\rangle$$

$$+ \int \frac{dk}{2\pi}\, \hbar\omega_k c_k |g, 1{:}k\rangle +$$

$$+ \hbar g \int \frac{dk}{2\pi} \underbrace{a_k^\dagger \sigma^- |e; vac\rangle}_{|g; 1{:}k\rangle} c_e + \hbar g \int \frac{dk}{2\pi}\, c_k |e; vac\rangle$$

$$i\hbar \dot{c}_e = \hbar g \int \frac{dk}{2\pi} c_k + \hbar \omega_e c_e$$

$$i\hbar \dot{c}_k = \hbar \omega_k c_k + \hbar g c_e$$

$$c_e(t=0) = 1 \qquad c_k(t=0) = 0$$

$$c_k(t) = -i \int_0^t dt' \, g c_e(t') e^{-i\omega_k(t-t')}$$

$$i\hbar \dot{c}_e(t) = g \int \frac{dk}{2\pi} \int_0^t (-ig) c_e(t') e^{-i\omega_k(t-t')} dt' + \hbar\omega_e c_e$$

$$\simeq -i\hbar g^2 \int_0^t dt' \, c_e(t') \int \frac{dk}{2\pi} e^{-i\omega_k(t-t')} + \hbar\omega_e c_e$$

Assume $\omega_k = ck$ (chiral, 1D mode)

$$\dot{c}_e(t) = -g^2 \int_0^t dt' \, c_e(t') \frac{\delta(t-t')}{c} - i\omega_e c_e \simeq -\underbrace{\frac{g^2}{2c}}_{\Gamma_e} c_e(t) - i\omega_e c_e$$

FGR $\quad \Gamma = \frac{2\pi}{\hbar} \int df \, |M_{fi}|^2 \, \delta(E_f - E_i)$

$$\Rightarrow \Gamma_e = \frac{2\pi}{\hbar} \int \frac{dk}{2\pi} \cdot |\hbar g|^2 \, \delta(\hbar ck - \hbar\omega_e)$$

$$= \frac{2\pi}{\hbar} \frac{1}{2\pi} |\hbar g|^2 \int d(\hbar ck) \, \delta(\hbar ck - \hbar\omega_e) \cdot \frac{1}{\hbar c} = \frac{g^2}{c} \quad ✓$$

$\cancel{2}$ pg 8

$$\dot{c}_e(t) = -\int_0^t dt'\, c_e(t') \cdot \underbrace{\int \frac{dk}{2\pi} e^{-i\omega_k(t-t')} |g_k|^2}_{G(t-t')} - i\omega_e c_e$$

WHITE bath $\longrightarrow$ $G(t-t') \propto \delta(t-t')$ Markov

general case $\longrightarrow$ memory kernel $\longrightarrow$ non-Markov

$$\tilde{G}(\omega) = \int d\tau\, e^{i\omega\tau} G(\tau) = \int d\tau\, e^{i\omega\tau} \int \frac{dk}{2\pi} e^{-i\omega_k\tau} |g_k|^2$$

$$= \int \frac{dk}{\cancel{2\pi}} |g_k|^2 \, \cancel{2\pi}\, \delta(\omega-\omega_k) = 2\pi \int d\omega_k \underbrace{\frac{1}{2\pi}\frac{dk}{d\omega_k}}_{\rho(\omega_k)} |g(\omega_k)|^2 \delta(\omega-\omega_k)$$

$$= 2\pi\, \rho(\omega)\, |g(\omega)|^2$$

density of states   matrix element

$$\dot{c}_e(t) = -i\omega_e c_e - \int_{-\infty}^{\infty} d\tau\, c_e(t-\tau) \cdot \Theta(\tau) \cdot G(\tau)$$

In Fourier $\quad -i\omega \tilde{c}_e = -i\omega_e \tilde{c}_e - R(\omega)\tilde{c}_e(\omega)$

$$R(\omega) = \int d\tau\, e^{-i\omega\tau}\, \Theta(\tau) G(\tau) = \int \frac{d\omega'}{2\pi} \tilde{\Theta}(\omega-\omega') \tilde{G}(\omega')$$

$$= \int \frac{d\omega'}{2\pi} \tilde{G}(\omega') \frac{-i}{\omega-\omega'+i0^+} = -i \, ⨍ \frac{d\omega'}{2\pi} \frac{G(\omega')}{\omega-\omega'} - i\left(\frac{i\pi}{2\pi}\right) G(\omega)$$

$$= -i \underbrace{⨍ \frac{d\omega'}{2\pi} \frac{G(\omega')}{\omega-\omega'}}_{\Delta_e(\omega)} + \frac{1}{2} G(\omega)$$

$$-i\omega\,\tilde{c}_e = -i(\underbrace{\omega_e + \Delta_e(\omega)}_{\text{Lamb shift}})\,\tilde{c}_e - \underbrace{\frac{1}{2}G(\omega)\,\tilde{c}_e}_{\text{decay}}$$

to be solved self-consistently in ω
or assuming $\Delta_e(\omega)$, $G(\omega)$ slowly varying in ω

$\hookrightarrow \partial_t c_e = -i(\omega_e + \Delta_e)\tilde{c}_e - \frac{G}{2} c_e$

<u>example</u> $\quad G(\omega) = 2\pi g^2 \dfrac{1}{2\pi} \dfrac{\gamma/2}{\left[(\omega-\omega_0)^2 + (\gamma/2)^2\right]}$ Lorentzian (e.g. damped cavity mode)

$$G(\tau) = 2\pi \int \frac{d\omega}{2\pi} e^{-i\omega\tau} G(\omega) = g^2 e^{-(\gamma/2 + i\omega_0)|\tau|}$$

$$\dot{c}_e(t) = -i\omega_e c_e - \int_0^t dt' \, g^2 e^{-\gamma(t-t')/2} e^{-i\omega_0(t-t')} c_e(t')$$

How to solve it?

$$\dot{b} = -i\omega_0 b - \frac{\gamma}{2} b + c_e \; ; \quad b(t=0) = 0$$

$$b(t) = \int_0^t dt' \, e^{-i\omega_0(t-t')} e^{-\gamma(t-t')/2} c_e(t')$$

$$\dot{c}_e(t) = -i\omega_e c_e - g^2 b \qquad 2\times \text{ODEs!}$$

$$\begin{pmatrix} \dot{c}_e \\ \dot{b} \end{pmatrix} = \begin{pmatrix} -i\omega_e & -g^2 \\ 1 & -i\omega_0 - \gamma/2 \end{pmatrix} \begin{pmatrix} c_e \\ b \end{pmatrix}$$

$$(-i\omega_e - \lambda)(-i\omega_0 - \gamma/2 - \lambda) + g^2 = 0$$

$$\lambda^2 + \lambda(i\omega_e + i\omega_0 + \gamma/2) + g^2 - \omega_e(\omega_0 + i\gamma/2)$$

$$\lambda = -\frac{1}{2}\left(i(\omega_e+\omega_0) + \frac{\gamma}{2}\right) \pm \frac{1}{2}\sqrt{\left(i(\omega_e+\omega_0)+\gamma/2\right)^2 - 4\left(g^2 - \omega_e(\omega_0+i\gamma/2)\right)}$$

$$= -i\,\frac{\omega_e+\omega_0}{2} - \frac{\gamma}{4} \pm \frac{1}{2}\sqrt{-(\omega_0 + i\gamma/2 - \omega_e)^2 - 4g^2}$$

i) Take $\omega_0 = \omega_e$:

$$= -i\omega_e - \frac{\gamma}{4} \pm \frac{1}{2}\sqrt{\frac{\gamma^2}{4} - 4g^2}$$

singuh in between

for $g \gg \gamma \longrightarrow \lambda = -i\omega_e - \frac{\gamma}{4} \pm g$ Rabi doublet

for $g \ll \gamma \longrightarrow \lambda = -i\omega_e - \gamma/2$ and

$$\lambda = -i\omega_e - \frac{\gamma}{4} + \frac{\gamma}{4}\sqrt{1 - 16 g^2/\gamma^2}$$

$$= -i\omega_e - \frac{2g^2}{\gamma}$$

$$\Gamma_e = \frac{2\pi}{\hbar}(\hbar g)^2 \cdot \frac{1}{2\pi} \cdot \frac{\gamma/2}{0 + (\gamma/2)^2} = 2g^2/\gamma$$ FGR ok for $g \ll \gamma/2$ *

FGR breaks down for $g \gtrsim \gamma/2$

ii) $\underbrace{|\omega_0 - \omega_e|}_{\delta} \gg g \gg \gamma$

$$\lambda = -i\,\frac{\omega_0+\omega_e}{2} - \frac{\gamma}{4} \pm \frac{i}{2}\sqrt{\delta^2 + i\delta\gamma - \cancel{\frac{\gamma^2}{4}} + 4g^2}$$

$$= -i\,\frac{\omega_0+\omega_e}{2} - \frac{\gamma}{4} \pm \frac{i\delta}{2}\left(1 + i\frac{\gamma}{2\delta} + \frac{2g^2}{\delta^2}\right)$$

$$\begin{cases} = -i\omega_0 - \dfrac{\gamma}{2} + i\dfrac{g^2}{\omega_e-\omega_0} = -i\left(\omega_0 + \dfrac{g^2}{\omega_0-\omega_e}\right) - \gamma/2 \\ = -i\left(\omega_e + \dfrac{g^2}{\omega_e-\omega_0}\right) \end{cases}$$

Lamb shift

* Zeno effect : start from $|e; no\,c\rangle$

$g \gg \gamma$ Rabi oscillations

$g \ll \gamma$ decay with rate $\frac{g^2}{\gamma}$. Measurement of b at rate γ blocks oscillations.

Effective decay $\sim \gamma^{-1}$.

[2-body losses in opt. lattice $\rightarrow$ TG/MI

Syassen et al. Science '08 $\rightarrow g^{(2)}$]

$$\rho_S(t) = |c_e|^2 |e\rangle\langle e| + \int \frac{dk}{2\pi} |c_k|^2 \, |g\rangle\langle g|$$ mixed state

but total system $\longrightarrow$ pure state

What at late time?

$$c_k(t) = -i \int_0^t dt' \, g \, c_e(t') \, e^{-i\omega_k (t-t')}$$

$$c_e(t') = e^{-i\omega_e t'} e^{-\Gamma_e t'/2}$$

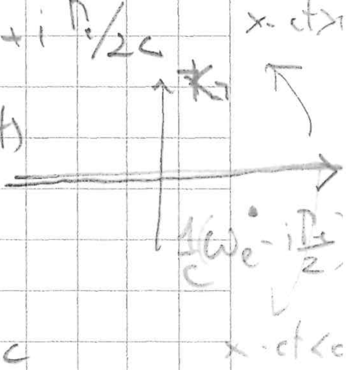
$$= -ig \int_0^t dt' \, e^{-i(\omega_e-\omega_k)t'} e^{-\Gamma_e t'/2} \cdot e^{-i\omega_k t}$$

$$\overset{\Gamma_e t \gg 1}{\simeq} -ig \frac{(-1)}{-i(\omega_e-\omega_k) - \Gamma_e/2} e^{-i\omega_k t}$$

$$= g \frac{e^{-i\omega_k t}}{\omega_k - \omega_e + i\Gamma_e/2}$$

Real space $$c(x,t) = \int \frac{dk}{2\pi} e^{ikx} \frac{g \, e^{-i\omega_k t}}{\omega_k - \omega_e + i\Gamma_e/2}$$

for $\omega_k = ck$ $$= \frac{g}{c} \int \frac{dk}{2\pi} e^{ik(x-ct)} \frac{1}{k - \frac{\omega_e}{c} + i\Gamma_e/2c}$$



$$= \Theta(ct-x) \left(-\frac{2\pi i g}{c}\right) \frac{1}{2\pi} e^{i\left(\frac{\omega_e}{c} - i\frac{\Gamma_e}{2c}\right)(x-ct)}$$

$$= \Theta(ct-x)\left(\frac{-ig}{c}\right) e^{i\frac{\omega_e}{c}x} e^{-i\omega_e t} e^{\Gamma_e(x-ct)/2c}$$

$c(x,t)$ = photon wave function.

<u>exercise</u> : extend to 2D/3D $\longrightarrow$ spherical wave

what happens for 2x emitters, spatially separated?

2-photon interference @ late times.

$\rightarrow$ How much quantum?

* Replace 2-LA with H.O, $|e\rangle \longrightarrow |1\rangle$

- same motion eqs.
- same equation for coherent state $|\alpha h : \alpha\rangle$

<u>Master equation</u>

$$\rho_S(t) \longrightarrow \rho_S(t'>t) = \mathcal{U}(t',t) \cdot \rho(t)$$

superoperator

Markov assumption

semi-group condition $\mathcal{U}(t_1,t_2)\,\mathcal{U}(t_2,t_3) = \mathcal{U}(t_1,t_3)$

trace conserved

$$\Rightarrow \text{Lindblad form } \frac{d\rho_S}{dt} = -\frac{i}{\hbar}[H_S, \rho_S] - \sum_k \left[A_k^\dagger A_k \rho_S + \rho_S A_k^\dagger A_k - 2 A_k \rho_S A_k^\dagger\right]$$

Microscopic derivation for Markovian bath, weak coupling

$\longrightarrow$ decay of $|e\rangle$

2-LA: $$\frac{d\rho_s}{dt} = -\frac{i}{\hbar}\left[H_s, \rho_s\right] + \frac{\Gamma_e}{2}\left[2\sigma^-\rho_s\sigma^+ - \sigma^+\sigma^-\rho_s - \rho_s\sigma^+\sigma^-\right]$$

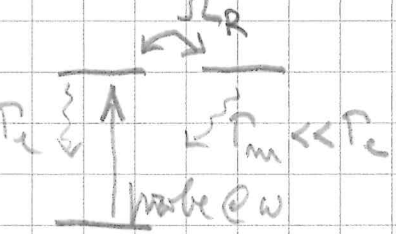
can also contain external driving

$\Longrightarrow$ Optical Bloch eqs. [IC's lect. notes on QO]

e.g. for weak monochromatic field $$\chi(\omega) = \frac{d_{eg}^2}{\omega_e - \omega - i\Gamma_e/2}$$

Lorentzian response like H.O.

3-LA:



$$i\dot{\alpha}_e = \omega_e\alpha_e - \frac{i\Gamma_e}{2}\alpha_e - \Omega_R\alpha_m + F$$

$$i\dot{\alpha}_m = \omega_m\alpha_m - i\frac{\Gamma_m}{2}\alpha_m - \Omega_R^*\alpha_e$$

$$\chi(\omega) = \frac{d_{eg}^2}{\omega_e - \omega - i\Gamma_e/2 - \dfrac{|\Omega_R|^2}{\omega_m - \omega - i\Gamma_m/2}}$$

What is absorption spectrum? (IC's QO lecture notes)

i) $|\Omega_R| \gg \Gamma_e, \Gamma_m$

$\longrightarrow$ polariton poles, for $\omega_e = \omega_m$ @ $\omega_e \pm |\Omega_R|$

line width $(\Gamma_e + \Gamma_m)/2$

Autler-Townes doublet

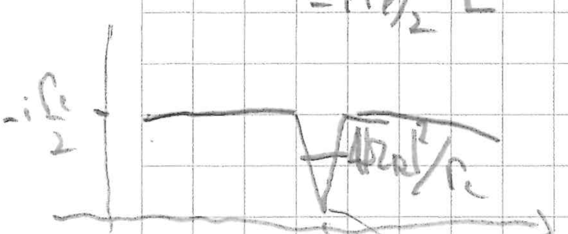
ii) $\Gamma_e \gtrsim |\Omega_R| \gg \Gamma_m$, look around ω_m

$$\chi(\omega) \simeq \frac{d_y^2\,(\omega_m - \omega - i\Gamma_m/2)}{(\omega_e - \omega - i\Gamma_e/2)(\omega_m - \omega - i\Gamma_m/2) - |\Omega_R|^2}$$

$$= \frac{d_y^2}{\omega_e - \omega - i\Gamma_e/2}\,\frac{(\omega_m - \omega - i\Gamma_m/2)}{\omega_m - \omega - i\Gamma_m/2 - |\Omega_R|^2/(\omega_e - \omega_m - i\Gamma_e/2)}$$

$$= \frac{d_y^2}{\omega_e - \omega - i\Gamma_e/2}\left[1 + \frac{|\Omega_R|^2}{\omega_e - \omega_m - i\Gamma_e/2}\,\frac{1}{\omega_m - \omega - i\frac{\Gamma_m}{2} - \frac{|\Omega_R|^2}{\omega_e - \omega_m - i\Gamma_e/2}}\right.$$

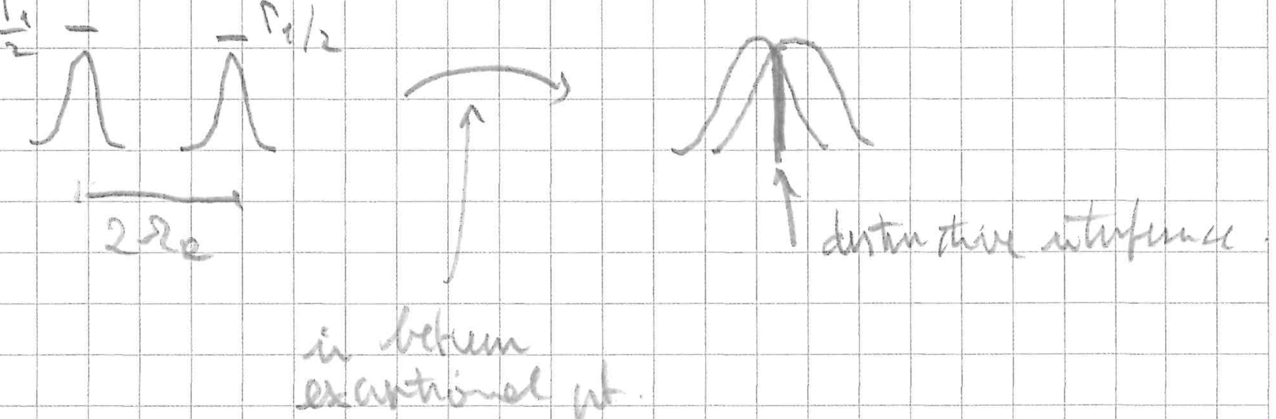
$$\overset{\omega_e = \omega_m}{\simeq} \frac{d_y^2}{-i\Gamma_e/2}\left[1 + \frac{|\Omega_R|^2}{-i\Gamma_e/2}\,\frac{1}{\omega_m - \omega - \frac{i}{2}\left(\Gamma_m + \frac{4|\Omega_R|^2}{\Gamma_e}\right)}\right]$$



$$\frac{d_y^2}{-i\Gamma_e/2}\left[1 - \frac{4|\Omega_R|^2}{\Gamma_e\left(\Gamma_m + \frac{4|\Omega_R|^2}{\Gamma_e}\right)}\right] \to 0 \quad \text{for } |\Omega_R|^2/\Gamma_e\Gamma_m$$

Narrow transparency dip $\longrightarrow$ slow light

atomic gas $\longrightarrow$ $v_g \simeq 20$ m/s.



Lindblad ME $\longrightarrow$ Monte Carlo wf

$$\frac{d\rho_s}{dt} = \frac{1}{i\hbar}[H,\rho_s] + \left(\Lambda\rho_s\Lambda^\dagger - \frac{1}{2}\Lambda^\dagger\Lambda\rho_s - \frac{1}{2}\rho_s\Lambda^\dagger\Lambda\right)$$

$$\Updownarrow$$

$$d|\psi\rangle = -\frac{i}{\hbar}H_s|\psi\rangle\, dt - \frac{1}{2}\Lambda^\dagger\Lambda|\psi\rangle\, dt$$ & renormalize

quantum jump $dp = \langle\psi|\Lambda^\dagger\Lambda|\psi\rangle\, dt$

2-LA : $\Lambda = \sqrt{\Gamma_e}\,\sigma_-$ jumps $|e\rangle \to |g\rangle$

HO : $\Lambda = \sqrt{\gamma}\, a$. On coherent state:

- jump has no effect

- $\omega = \omega_0 - i\gamma/2$ on determ. evol.

On cat state $\left(|coh:\alpha\rangle + |coh:-\alpha\rangle\right)/\sqrt{2}$

$$\Lambda|cat\rangle = \frac{\alpha}{\sqrt{2}}\left(|coh:\alpha\rangle - |coh:-\alpha\rangle\right)$$

switch sign

1 jump destroys coherence

$$dp/dt = \gamma\langle a^\dagger a\rangle = \gamma\langle N\rangle$$

grows fast in classical regime.

3-LA : $\Lambda_e = \sqrt{\Gamma_e}\,|g\rangle\langle e|$; $\left(\Lambda_m = \sqrt{\Gamma_m}\,|g\rangle\langle m| \text{ negligible}\right)$

in coupled basis $|\pm\rangle = \frac{1}{\sqrt{2}}\left(|e\rangle \pm |m\rangle\right)$ of eigenstates of H_R

$$\Lambda_e = \sqrt{\Gamma_e}\,|g\rangle\left(\frac{\langle +| + \langle -|}{\sqrt{2}}\right)$$ sensitive to coherence of $\pm$ states

NOTE: other unravelings of Lindblad $\to$ MCWF are possible $\to$ correspond to other measurement schemes.

non-Markov effects

$$H = \omega_c a^\dagger a + \omega_e \sigma^+\sigma^- + g\,\sigma_x \cdot (a+a^\dagger)$$

+ radiative coupling $\approx \int \frac{dk}{2\pi} (a+a^\dagger)(b_k + b_k^\dagger)$ + spin losses $\approx \int \frac{dk}{2\pi} \sigma_x (c_k + c_k^\dagger)$

Naive RWA $\sim \int \frac{dk}{2\pi} (a b_k^\dagger + a^\dagger b_k)$ + white bath

gives jump operators $\hat{a}$ and $\hat{\sigma}^-$

H in RWA $\rightarrow$ Jaynes-Cummings ladder

$|gs\rangle = |g;0\rangle$, jumps go down

beyond RWA $\rightarrow H_{\text{anti-RWA}} = g(a\sigma^- + a^\dagger\sigma^+)$

mix $|g\,0\rangle$ with excited states $N=2$

$$|gs\rangle = |g,0\rangle + O\left(\frac{g}{\omega_{c,e}}\right)\underbrace{|e,1\rangle}_{} + \dots$$

allows for jumps to $|g,1\rangle, |e,0\rangle$
that are (mostly) excited states

Wrong assumption: white bath, with $DOS(E<0) > 0$.

IC, Ciuti PRB 2005 ; PRA 2006
(+ Bastard)

$\hookrightarrow$ Input-Output approach

Redfield eq: $\mathcal{L}(\rho) = \Gamma\left[U\rho S + S\rho U^\dagger - SU\rho - \rho U^\dagger S\right]$

S = coupling, e.g. $\sigma^- + \sigma^+$ or $a + a^\dagger$

reduces to S in white approx $\longleftarrow$ $U = \int_0^\infty d\tau\, \nu(\tau)\, e^{-iH\tau} S e^{iH\tau}$

$$U_{ij} = \tilde{\nu}(\omega) \times \tilde{\Theta}(\omega)\Big|_{\omega_j - \omega_i} S_{ij}$$

$$= \left(-i\, PP\frac{1}{\omega} + \pi\delta(\omega)\right) \times \tilde{\nu}(\omega)\Big|_{\omega_j-\omega_i} S_{ij}$$

eigenstates of H

$$= \pi\, \tilde{\nu}(\omega_j - \omega_i) - i\, PP\frac{1}{\omega} \times \tilde{\nu}(\omega) \quad S_{ij}$$

jump process weighted by $\tilde{\nu}(\omega_j - \omega_i)$ — Lamb shift

Modulo some subtleties (B-P book): $\langle i|\Lambda|j\rangle \longrightarrow \langle i|\bar{\Lambda}|j\rangle$

$$= \langle i|\Lambda|j\rangle \sqrt{\tilde{\nu}(\omega_j - \omega_i)}$$

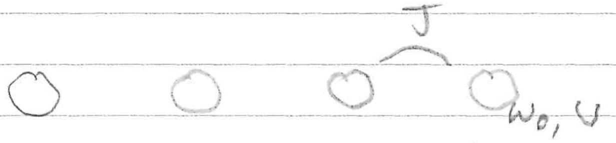
weighted jump operator

* kills spurious processes that increase energy
* requires preliminary diag of H; assumes weak coupling

More sophisticated forms of ME $\longrightarrow$ Schnell, Eckardt

losses small perturbation on top of conservative evol $\longrightarrow$ problems for spatially extended systems with dense spectrum.

Biella / Lebreuilly / Caleffi model 2015 → 2023
2x PRA'17
1x PRL'23.
CRAS'16



lattice of cavities ω_0, hopping J, nonlinearity U → goal photon MI

$$H = \sum_i \omega_0 a_i^\dagger a_i + \frac{U}{2} a_i^\dagger a_i^\dagger a_i a_i - J \sum_{\langle ij \rangle} a_i^\dagger a_j$$

radiative losses Γ_c : $\Lambda_r = \sqrt{\Gamma_c}\, a_i$

pumping P : Markov $\Lambda_p = \sqrt{P}\, a_i^\dagger$
↳ insensitive to U
steady-state $T = \infty$.

non-Markov : weighted $\tilde{\Lambda}_p$ so that
double occupancy forbidden

saturation via $\Lambda_2 = \sqrt{\Gamma_2}\, a_i^2$
(2-photon losses)

or microscopic model : each site 2-LA, pumped
to popul. inversion

– gain centred @ ω_e, width Γ_e

– gain saturation

At MF $\langle a \rangle$, $\langle \sigma_z \rangle$, $\langle \sigma^+ \rangle$ → Lamb semiclassical
theory of laser

Here for $U \gg$ → quantum features, photon MI

[lc, CRAS 2025]