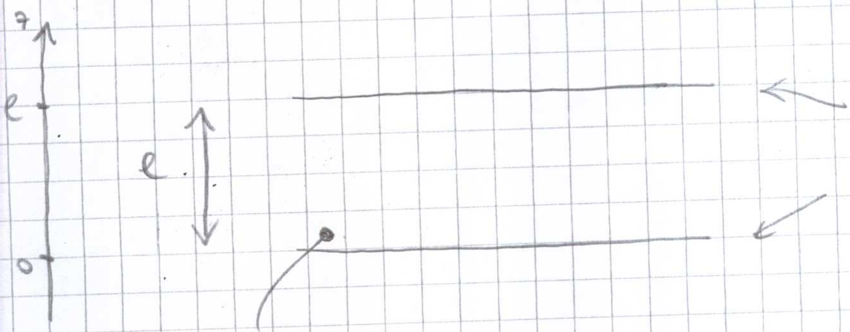
Lecture 2

Planar optical cavities (neglect for simplicity polarisation)



metallic mirrors → $E(r_{\parallel}, z=0, \ell) = 0$

refractive index n_0

$$E(r_{\parallel}, z, t) = \sum_{n} \sum_{k_{\parallel}} e^{i k_{\parallel} \cdot r_{\parallel}} \sin\left(\frac{\pi n z}{\ell}\right) \cdot \mathcal{E}_{m}(k_{\parallel})\, e^{-i \omega_m(k_{\parallel}) t}$$

in-plane wavevector

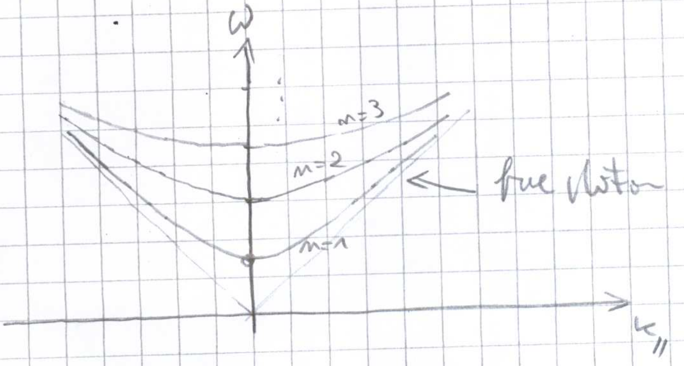
$1 < n =$ mode index along z

mode amplitude

time evolution

frequency $\omega_m(k_{\parallel}) = \sqrt{\frac{c^2}{n_0^2}\left[\left(\frac{\pi m}{\ell}\right)^2 + k_{\parallel}^2\right]}$

"dispersion law"



free photon in n_0 medium $\omega = \frac{c}{n_0}|k|$

$$\omega_m(k_{\parallel}) \simeq \underbrace{\frac{\pi m c}{n_0 \ell}}_{\omega_0} + \underbrace{\frac{e}{2 n_0} \frac{\ell}{\pi m} k_{\parallel}^2}_{} = \omega_0 + \frac{\hbar}{2 m^*} k_{\parallel}^2 + \ldots$$

$m^* = \frac{\pi m n_0}{c \ell} \hbar$ "effective mass of photon"

NOTE: cut-off $\hbar\omega_0 = m^*\left(\frac{c}{n}\right)^2$!!!

Restrict to single mode

$$E(r_\parallel, z, t) = \sin\left(\frac{\pi z}{\ell}\right) \int \frac{d^2k_\parallel}{(2\pi)^2}\, e^{i k_\parallel r_\parallel}\, \underbrace{\mathcal{E}(k_\parallel)\, e^{-i\omega(k_\parallel)t}}_{\mathcal{E}(k_\parallel, t)}$$

motion eq. $$\frac{\partial}{\partial t}\mathcal{E}(k_\parallel, t) = -i\,\omega(k_\parallel)\,\mathcal{E}(k_\parallel, t)$$

$$\simeq -i\left[\omega_0 + \frac{\hbar}{2m^*}k_\parallel^2 + \dots\right]\mathcal{E}(k_\parallel, t)$$

after Fourier transform to $r_\parallel$:

$$\frac{\partial}{\partial t}E(r_\parallel, t) = -i\left[\omega_0 - \frac{\hbar}{2m^*}\nabla^2_{r_\parallel} + \dots\right]E(r_\parallel, t)$$

Schrödinger-like eq. for massive photon

↳ Similar to paraxial eqs for light propagation ...

↳ ADDENDUM on STRONG COUPLING

Slow spatial modulation of $n_0(r_\parallel)$:

$$\therefore\quad i\frac{\partial}{\partial t}E(r_\parallel, t) = \left[\underbrace{\omega_0(r_\parallel)} - \frac{\hbar}{2m^*}\nabla^2\right]E(r_\parallel, t)$$

same role as external potential in Schrödinger eq

→ must not couple states of different n.

i.e. $\frac{\hbar^2}{2m^* L^2} \ll \hbar\omega_0$.

Amplifying / Lossy medium :

$$n_0^2 = \varepsilon = \varepsilon_r + i\,\varepsilon_i = 1 + 4\pi\,(\chi_r + i\,\chi_i)$$

$$n_0 \simeq \underbrace{1 + 2\pi(\chi_r}_{\text{real part}} + i\,\chi_i)$$

↖ gives imaginary correction to $\omega(r_\parallel)$

$$\omega_0 \simeq \frac{\pi m c}{n_0 \ell} \simeq \frac{\pi m c}{n_0^R \ell}\left(1 - i\,\frac{n_0^I}{n_0^R}\right) = \underbrace{\frac{\pi m c}{n_0^R \ell}}_{\omega_0^R} - i\,\underbrace{\frac{\pi m c}{(n_0^R)^2 \ell}\, n_0^I}_{\gamma_0/2}$$

$$\begin{cases} \text{damping } (n_0^I > 0) \\ \text{amplification } (n_0^I < 0) \end{cases}$$

Optical nonlinearity $n_0 = n_0(I) \simeq n_0^{(0)} + n_0^{(3)} \cdot |E|^2$

$$i\,\frac{\partial}{\partial t} E(r_\parallel, t) = \left[\omega_0(r_\parallel) - \frac{\hbar}{2m^*}\nabla^2 - \frac{\pi m c}{n_0^2 \ell}\, n_0^{(3)} |E|^2\right] E$$

Nonlinear Schrödinger equation.

Photons see an "effective potential" due to interactions

$$V_{int}(r_\parallel) = -\frac{\hbar\pi m c}{n_0^2 \ell}\, n_0^{(3)} |E|^2$$

$$\begin{cases} n_0^{(2)} < 0 \longrightarrow \text{attractive "focussing" nonlinearity} \\ n_0^{(3)} > 0 \longrightarrow \text{repulsive "defocussing" nonlinearity} \end{cases}$$

↳ Classical field equation derived from Maxwell's eqs in nonlinear material medium.

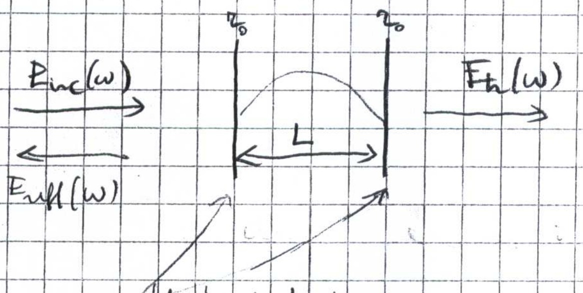
Upon quantization : $E(r_\parallel) \longrightarrow \hat{E}(r_\parallel)$

with commutation relations

$$\left[\hat{E}(r_\parallel),\ \hat{E}^\dagger(r'_\parallel)\right] = \delta(r_\parallel - r'_\parallel) \cdot \underset{\substack{\uparrow \\ \text{suitable} \\ \text{normalization factor}}}{\#}$$

$$H = \underset{\substack{| \\ \text{normaliz. factor related} \\ \text{to e.m. field energy}}}{\boxed{\#}} \int d^2 r\ \hat{E}^\dagger(r_\parallel) \left[\hbar\omega_0(r_\parallel) - \frac{\hbar^2}{2m^*}\nabla^2 + g\,\hat{E}^\dagger(r_\parallel)\hat{E}(r_\parallel)\right]\hat{E}(r_\parallel)$$

Coupling to external world via non-perfectly reflecting mirrors



mirror reflectivity $|r_0| < 1$

theory of F-P cavity :

$$t(\omega) = \frac{E_{tr}(\omega)}{E_{inc}(\omega)} = \frac{t_0^2\, e^{i\omega L/c}}{1 - r_0^2\, e^{2i\omega L/c}} \simeq t_0^2\, e^{i\omega L/c} \cdot \sum_n \frac{1}{1 - r_0^2 - 2i(\omega - \omega_n)\frac{L}{c} r_0^2}$$

$$\overset{\text{single mode}}{=} e^{i\omega L/c} \frac{i\frac{c}{2L}T}{\omega - \omega_0 + i\frac{cT}{2L}} = e^{i\omega L/c} \frac{i\,\Gamma/2}{\omega - \omega_0 + i\,\Gamma/2}$$

with $\Gamma = \frac{cT}{L}$, $T = |t_0|^2$ radiative loss rate of cavity in-cavity field $E_{cav}(z) \simeq E_{cav} \sin\left(\frac{\pi z}{L}\right)$, $z = 0 \rightarrow L$

$$E_{cav}(\omega) = \frac{E_{tr}(\omega)}{t_0} = \frac{i \; ct/2L}{\omega - \omega_0 + i\,\Gamma/2} \; E_{inc}(\omega)$$

$$\left(\omega - \omega_0 + i\,\Gamma/2\right) E_{cav}(\omega) = i \frac{ct_0}{2L} E_{inc}(\omega)$$

after Fourier to time:

$$\left(i\partial_t - \omega_0 + i\,\Gamma/2\right) E_{cav}(t) = i \frac{ct_0}{2L} E_{inc}(t)$$

$$\partial_t E_{cav}(t) = \left(-i\omega_0 - \frac{\Gamma}{2}\right) E_{cav}(t) + \frac{ct_0}{2L} E_{inc}(t)$$

very similar to Schrödinger eq.
(all other terms easily added ...)

radiative loss term $\Gamma = \frac{cT}{L}$ decay through non-perfect mirrors.

external driving force
(inhomogeneous term!)

$$H_{drive} = \int d^2 r_\parallel \left[\cancel{t_s} \cdot E_{inc}(r_\parallel, t) \underbrace{\hat{E}^\dagger(r_\parallel)}_{} + h.c. \right]$$

one-operator term!

__Loss term__ does not conserve phase-space volume → not included in Hamiltonian formalism for isolated cavity at photon level

↳ MASTER eq. description

Equation of motion for cavity field

$$i \frac{\partial}{\partial t} E_{cav}(r_{\parallel}, t) = \left[\omega_0 - i\frac{\Gamma}{2} - \frac{\hbar}{2m^*}\nabla^2 + \delta\omega(r_{\parallel}) + g_{NL}|E|^2\right] E +$$

$$+ i \frac{c t_0}{2L} E_{inc}(r_{\parallel}, t)$$

and $E_{tr}(r_{\parallel}, t) = t_0 E_{cav}(r_{\parallel}, t)$

driven-dissipative motion equation for cavity field

"driven-dissipative Gross-Pitaevskii eq. for condensate of photons in cavity"

↳ it used to describe the evolution of coherent state $|coh : E_{cav}(r_{\parallel}, t)\rangle$

Bibliography: lecture notes on "Quantum Optics" @ www.science.unitn.it/~carusott

Addendum : Strong light-matter coupling

cavity mode coupled to resonant matter excitation

$P(r,t)$ = matter polarization

$$\partial_t P = -i\omega_x P - i\Omega_R E - \frac{\Gamma_x}{2} P$$

light-matter coupling term

dissipation term.

$$\partial_t E = -i\left[\omega_0 - \frac{\hbar}{2m^*}\nabla^2 + \Omega_R P - i\frac{\Gamma}{2}\right] E + \ldots$$

↳ can be derived from elastically bound electron model or more refined quantum descriptions of excitonic transitions in solids.

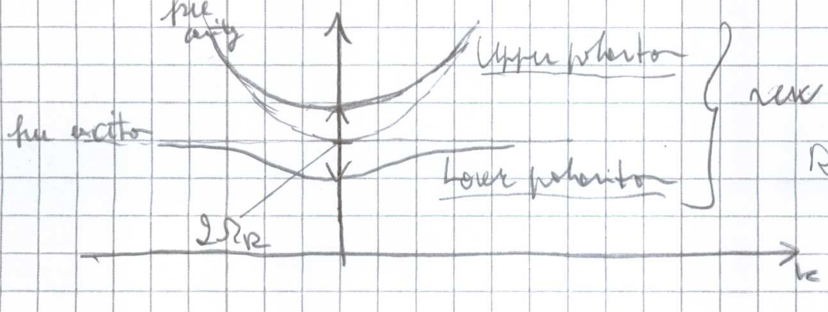
Coupled modes

$$i\partial_t \begin{pmatrix} E \\ P \end{pmatrix} = \underbrace{\begin{pmatrix} \omega_0 - \frac{\hbar}{2m}\nabla^2 & \Omega_R \\ \Omega_R & \omega_x \end{pmatrix}}_{M} \begin{pmatrix} E \\ P \end{pmatrix}$$

at given k :

$$M = \begin{pmatrix} \omega_0 + \frac{\hbar k^2}{2m} & \Omega_R \\ \Omega_R & \omega_x \end{pmatrix}$$

Resonant case $\omega_0 = \omega_x$:



new coupled modes.

Resolved if $\Omega_R > \Gamma_P, \Gamma$

STRONG COUPLING