(Not so) quantum ducks — sketch of solution

1) generic wave equation in homogeneous geometry:

$$G(k,\omega)\, h(k,\omega) = \mathcal{F}(k,\omega)$$

$\mathcal{F}$ → applied force.

G → satisfies $G(k,\omega(k)) = 0$. or dispersion law

In this vicinity $G(k,\omega) \simeq \underbrace{G(k,\omega(k))}_{0} + (\omega-\omega(k))\, G'(k,\omega)$

$$\Longrightarrow \quad G'(k,\omega(k))(\omega-\omega(k))\, h(k,\omega) = \mathcal{F}(k,\omega)$$

Back to time-domain $\quad \omega \longrightarrow +i\frac{\partial}{\partial t}$

$$\left(i\frac{\partial}{\partial t} - \omega(k)\right) h(k,t) = \underbrace{\frac{\mathcal{F}(k,t)}{G'(k,\omega(k))}}_{\text{renormalised force } f(k,t)}$$

2) Uniform motion implies $f(r,t) = f(r-vt)$, so:

$$f(k,\omega) = \int d^3r \int dt\, e^{-ikr}\, e^{i\omega t}\, f(r-vt) =$$

$$= \int d^3r\, e^{-ik(r-vt)}\, f(r-vt) \underbrace{\int dt\, e^{-ikvt}\, e^{i\omega t}}_{2\pi\, \delta(\omega - k\cdot v)} =$$

$$= \underbrace{\int d^3r\, e^{-ikr} f(r)}_{f(k)} \quad 2\pi\, \delta(\omega - k\cdot v)$$

Massive particle (e.g. duck) : exchange Δk, $\Delta\omega$ into wave

$P \to P - \hbar\Delta k \qquad E_k \to E_k - \hbar\Delta\omega$

$$\frac{P^2}{2m} \longrightarrow \frac{(P-\hbar\Delta k)^2}{2m} = \frac{P^2}{2m} - \frac{P}{m}\hbar\Delta k + \underbrace{\frac{\hbar^2\Delta k^2}{2m}}_{\text{small}}$$

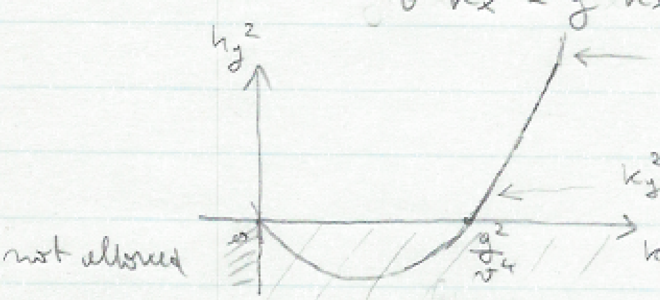
$$\simeq \frac{P^2}{2m} - v\cdot\hbar\Delta k = E_k - \hbar\Delta\omega$$

$\Rightarrow \Delta\omega = \Delta k\cdot v$ relate energy and momentum of emitted wave.

3+4) $\omega(k) - k\cdot v = 0 \Rightarrow \sqrt{gk} = k\cdot v = v k_x$

$$g\sqrt{k_x^2+k_y^2} = v^2k_x^2 \qquad \text{(only for } k_x \geq 0\text{)}$$

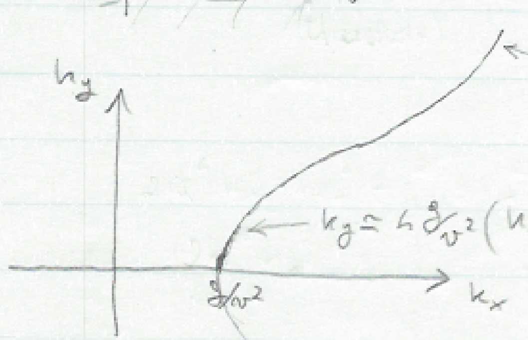
$$v^4k_x^4 - g^2k_x^2 = g^2k_y^2$$



$k_y^2 \simeq \frac{v^4}{g^2}k_x^4$ (for $k_x^2 \gg g^2/v^4$)

$k_y^2 \simeq 2(k_x^2 - \frac{g^2}{v^4}) = 2(k_x + \frac{g}{v^2})(k_x - \frac{g}{v^2})$

$\simeq 4\frac{g}{v^2}(k_x - \frac{g}{v^2})$



$k_y \simeq \frac{v^2}{g}k_x^2$

$k_y \simeq 4\,g/v^2\,(k_x - g/v^2)$

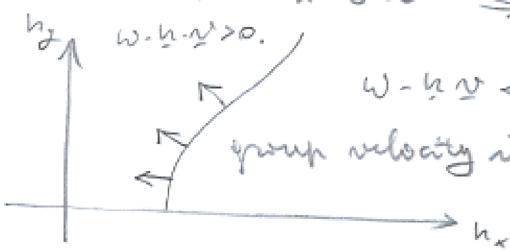
In lab frame $v_g = D_k \omega$.

In duck frame $v_g' = D_k \omega - v = D_k(\omega - k \cdot v)$,

which is orthogonal to set $\omega - \underline{k}\cdot\underline{v} = 0$

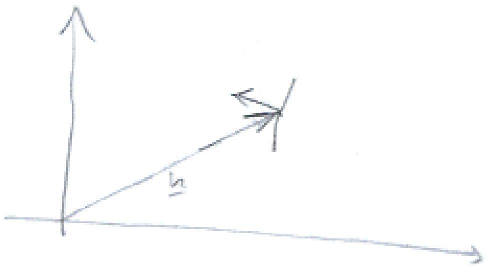
To determine direction:

for $k_x \to \infty$ $\Rightarrow$ $\sqrt{gk} < v k_x$



Roles of phase and group velocities explained in 1202.3494

In brief:



$k \in$ locus of $\omega - \underline{k}\,\underline{v} = 0$ $\longrightarrow$ fringe pattern $\perp$ $\underline{k}$

$\longrightarrow$ extends in direction $\underline{v}_g$ from duck

Angle of v_g determined by $\dfrac{dk_y}{dk_x} = \dfrac{d}{dk_x}\dfrac{1}{g}\sqrt{v^4 k_x^4 - g^2 k_x^2} = \dfrac{2v^4 k_x^3 - g^2 k_x}{g\sqrt{v^4 k_x^4 - g^2 k_x^2}}$

$\min\left(\dfrac{dk_y}{dk_x}\right)$ is at $k_x^2 = \dfrac{3g^2}{2v^4}$

equal to $\sqrt{8}$.

$$\underline{v}_g \perp \left(1, \frac{dk_y}{dk_x}\right) \Rightarrow (v_g)_y = -\frac{1}{dk_y/dk_x}(v_g)_x$$

$\max(\tan\theta_g) = \dfrac{v_{gy}}{v_{gx}} = -\dfrac{1}{\sqrt{8}}$, i.e. $\max\theta_g = 19.5°$

In general:

for $\omega(k) \sim |k|^z$

rescale $v \to \alpha v$

locus of $(\omega - k v) = 0$ is invariant

if $k \to \alpha^\beta k$ with $\beta z = \beta + 1$,

i.e. $\beta = \frac{1}{z-1}$

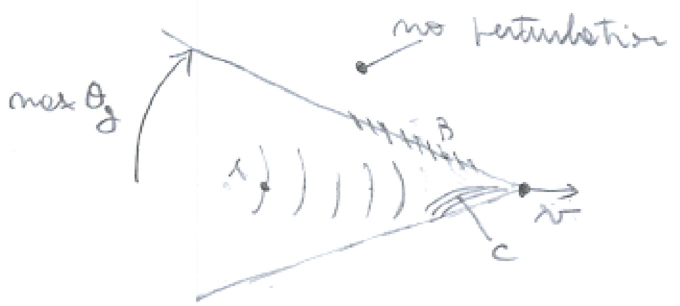
For our $z = 1/2$ case $\longrightarrow \beta = -2$

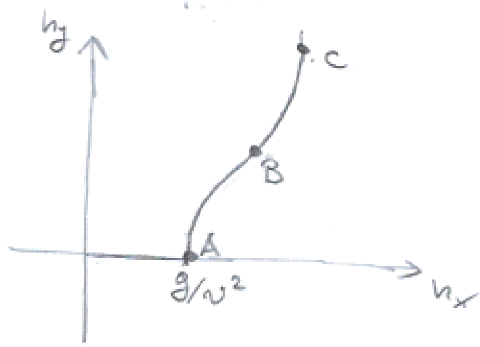
(corresponds to threshold $k_x|_{th} = g/v^2$)

$\Longrightarrow$ rescaling of k does not change emission angle and fringe pattern, just spatial scale $r \to \alpha^\beta r$

For $z=1 \to$ no solution for $\beta \longrightarrow$ wake angle varies with v

5) Shape of wake





From picture $\lambda_A \sim$ size duck $\sim 30\,cm = l_D$

$2\pi/(g/v^2)$ $\Longrightarrow \frac{2\pi v^2}{g} = l_D \Longrightarrow v = \left(\frac{g\, l_D}{2\pi}\right)^{1/2} \simeq 0{,}6\, m/s$

(reasonable!)