From coupled pendula to Electromagnetically Induced Transparency

Consider a pair of identical pendula mechanically coupled by a spring. Both pendula consist of a point-like mass m attached to a massless wire of length ℓ. The spring has a spring constant κ and a rest length equal to the distance of the suspension points of the pendula. The equilibrium position of the system therefore consists of the two pendula being vertically oriented. Friction effects act on the pendula with damping rates $\gamma_{1,2}$. The second pendulum is driven by a time-dependent external force $F_{ext}(t)$.

1. Write the equation of motion for the displacements $x_{1,2}$ and the momenta $p_{1,2}$ of the two masses in the small oscillation limit.

2. Write the equations of motion for the harmonic oscillator amplitudes

$$\alpha_{1,2} = \sqrt{\frac{m\omega_0}{2\hbar}}\, x_{1,2} + \frac{i}{\sqrt{2m\hbar\omega_0}}\, p_{1,2}.$$

   Assuming that the bare pendulum frequency $\omega_0 = \sqrt{g/\ell}$ (g is here the gravitational acceleration) is the fastest time scale in the problem, perform a rotating wave approximation in the equations of motion for $\alpha_{1,2}$.

3. For a monochromatic driving force at a frequency ω in the vicinity of ω_0 (such that $|\omega - \omega_0| \ll \omega_0$), show that the oscillation amplitude of the second mass can be written in the form

$$\bar{\alpha}_2 = \frac{f^o_{ext}}{-\delta - i\gamma_2/2 - \frac{\Omega^2}{-\delta - i\gamma_1/2}},$$

   where we have set $f^0_{ext} = F_{ext}/\sqrt{2m\hbar\omega_0}$ and $\Omega = \kappa/2m\omega_0$ and $\delta = \omega - \Omega - \omega_0$.

4. Give an expression for the poles of the response in the two limits (i) $\Omega \gg \gamma_{1,2}$ and (ii) $\gamma_1 \gg \Omega \gg \gamma_2$. Give an expression for the corresponding residues.

5. Using the results of the previous point, provide a schematic plot of the behaviour of the absorbed energy

$$W = \frac{1}{2}\,\mathrm{Re}\left[f^{o*}_{ext}\,(-i\omega\bar{\alpha}_2)\right]$$

   as a function of the driving frequency ω in the two limits.

6. Provide a physical interpretation of the lineshape in terms of Autler-Townes splitting of an atomic resonance line in the limit (i) and of Gozzini-Alzetta Electromagnetically Induced Transparency effect in the limit (ii).