The gapped Goldstone mode of a coherent injected laser

We model an externally injected laser in a spatially extended planar geometry by means of a classical field equation for the in-cavity field $E(\mathbf{r},t)$ in the form:

$$i\frac{\partial E}{\partial t} = \omega_0\, E - \frac{\hbar}{2m^*}\,\nabla^2 E + g\,|E|^2\,E + \frac{i}{2}\left(\frac{P}{1+|E|^2/n_s} - \gamma\right)\,E + iE_{\mathrm{inc}}\,e^{-i\omega_{\mathrm{inc}}t}. \qquad (1)$$

The cut-off frequency of the planar cavity is ω_0, the effective photon mass is m^* and g is a nonlinear interaction constant describing the blue-shift of the optical mode due to a $\chi^{(3)}$ susceptibility of the cavity material. Furthermore, γ is the linear loss rate, P is the pump strength and n_s the gain saturation density. The spatial derivatives are meant to be taken along the $\mathbf{r} = \{x, y\}$ in-cavity directions only.

The initial steps of the study are very similar to the Exercise 4 of the Quantum Optics course, *The collective excitation modes of a lasing device.* As a key step forward, we now focus on the effect of a monochromatic incident field at normal incidence. This is described in our model by the last term in the equation, namely a coherent drive of spatially constant amplitude E_{inc} and frequency ω_{inc}.

In the first part of the Research Exercise, we aim at characterizing the dynamical evolution of a spatially uniform field, in particular its steady-states and possible limit cycle behaviours.

1. Write the evolution equation of the field $\bar{E}$ in the rotating frame at ω_{inc} where no explicit time-dependence is present in the equation of motion for the field.

2. Assuming that the field is spatially uniform along the $\mathbf{r}$ plane, write an ordinary differential equation for its amplitude and, then, an algebraic equation for its steady-state amplitude $\bar{E}_{ss}$.

   (a) Assume for this question $g = 0$ and $\omega_{\mathrm{inc}} - \omega_0 = 0$. Keeping all other parameters fixed, write the incident intensity $|E_{\mathrm{inc}}|^2$ as a function of the steady-state amplitude $|\bar{E}_{ss}|^2$. Identify the different regimes in analogy to the magnetization of a ferromagnet under an external magnetic field. Discuss the relation between the phase of $\bar{E}_{ss}$ and the one of E_{inc}.

   (b) Assume for this question $g = 0$. Study the incident intensity $|E_{\mathrm{inc}}|^2$ as a function of the steady-state amplitude $|\bar{E}_{ss}|^2$ and discuss the relation between the phase of $\bar{E}_{ss}$ and the one of E_{inc}. Clarify the physical role of the detuning $\omega_{\mathrm{inc}} - \omega_0$.

   (c) Repeat the study for non-zero g and identify regimes displaying multi-stability phenomena.

3. Keep assuming that the field amplitude is spatially uniform. Make use of some numerical plotting software to display the vector field corresponding to the time-evolution of the (spatially uniform) amplitude $\bar{E}$ in the complex plane. Perform a qualitative study of the main features of this vector field for different choices of the parameters.

    (a) Identify within this formalism the steady-states found above and, making use of the plots, try to characterize their dynamical stability in the different regimes. Give a physical interpretation of the results.

    (b) Using again a combination of a numerical software and of qualitative methods for differential equations, try to identify other solutions of the time-evolution of the field, in particular in the form of a limit cycle.

    (c) Show that such limit cycle solutions are in particular possible in the regime of small $E_{\rm inc}$ and non-zero $\omega_{\rm inc} - \omega_0$. Give a physical interpretation of these limit cycle solutions and, in particular, of their period.

In the second part of the Research Exercise, we aim at studying the collective excitation modes around the different steady states. In particular, we wish to understand the nature of the gap that opens in the Bogoliubov dispersion relation as a consequence of the explicitly-symmetry-breaking term $E_{\rm inc}$. The general approach closely follows the calculations of Exercise 4.

4. Write the partial differential equations describing the linearized time-evolution of a small perturbation $\delta\bar{E}(\mathbf{r},t)$ around a steady state $\bar{E}_{ss}$. Taking advantage of the translational invariance of the problem, use the complex basis $(\delta\bar{E}_{\mathbf{k}}, \delta\bar{E}^*_{-\mathbf{k}})$ and reduce the partial differential equations to a 2×2 eigenvalue problem for the dispersion relation $\omega(\mathbf{k})$.

5. For each of the different steady states found in the previous questions, study the shape of the dispersion relation $\omega(\mathbf{k})$ in the different regimes of parameters.

6. In particular, characterize in which cases the gap in the collective excitation spectrum opens in the real and/or in the imaginary part of $\omega(\mathbf{k}=0)$. Make use of this theory to assess the dynamical stability or instability of the different steady-states.

7. Speculate on the shape of the collective excitation spectrum around a limit cycle solution, especially in the limit of small $E_{\rm inc}$.

A presentation of the general theory underlying this Research Exercise can be found in Secs.IV-B1 & C and Secs.VI-A1 & B of Rev. Mod. Phys. **85**, 299 (2013).
The general goal of the exercise is to provide theoretical understanding of the experimantal observations in arXiv:2310.11903 (to appear in Nature Physics, 2025).