The collective excitation modes of a VCSEL planar laser

A generic model of non-equilibrium condensation can be obtained starting from the semiclassical Lamb-Haken theory of laser. In a spatially extended planar geometry, this approach leads to a classical field equation of the form:

$$i\frac{\partial E}{\partial t} = \omega_0 \, E - \frac{\hbar}{2m^*} \nabla^2 E + g \, |E|^2 \, E + i \left(\frac{P_0}{1 + \beta |E|^2} - \gamma \right) \, E, \tag{1}$$

where ω_0 is the cut-off frequency of the planar cavity, m^* is the effective photon mass, γ is the linear loss rate, the linear gain rate P_0 is proportional to the pump strength and the β coefficient quantifies the gain saturation. The nonlinear interaction constant g describes the blue-shift of the optical mode due to a $\chi^{(3)}$ susceptibility of the cavity material. The spatial derivatives are meant to be taken along the xy cavity plane only, while the field along z is to be considered as frozen into the lowest mode.

All parameters will be kept fixed but for the pump parameter P_0 which is spanned across the condensation threshold at $P_0 = \gamma$.

1. Assuming that lasing occurs in a plane-wave mode at $k = 0$, evaluate the steady-state value E_0 of the laser field in the cavity as a function of P_0 and identify its singularity at the threshold.

2. Provide a physical explanation why lasing into the $k = 0$ mode should be favoured in a real device? Identify regimes where this is not the case.

3. By linearizing the field equation around the steady-state solution, study the behavior of the effective damping rate of fluctuations as the pump strength P_0 tends to the threshold from below, $P_0 \to \gamma^-$. Interpret this effect in terms of a critical slowing down phenomenon.

4. For a pump strength above the lasing threshold $P_0 > \gamma$, determine the dispersion of the collective excitations around the stationary state, that is the frequency $\omega_{\mathbf{k}}$ of a perturbation on top of the spatially homogeneous lasing configuration as a function of its wavevector $\mathbf{k}$.

   (a) As a first step, show that the linearized evolution of the field around the spatially homogeneous steady state has the form:

   $$i\frac{d}{dt}\begin{pmatrix} \delta E_{\mathbf{k}} \\ \delta E^*_{-\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \omega_0 + \frac{\hbar k^2}{2m} + (g - iP_1)|E_0|^2 & (g - iP_1)E_0^2 \\ -(g + iP_1)E_0^{*2} & -\omega_0 - \frac{\hbar k^2}{2m} - (g + iP_1)|E_0|^2 \end{pmatrix} \begin{pmatrix} \delta E_{\mathbf{k}} \\ \delta E^*_{-\mathbf{k}} \end{pmatrix}$$

   where the field fluctuations are described in the $(\delta E_{\mathbf{k}}, \delta E^*_{-\mathbf{k}})$ Fourier basis where $\delta E_{\mathbf{k}}$ is the Fourier transform of $\delta E(\mathbf{r})$. Express the gain saturation coefficient P_1 in terms of the parameters in Eq.1 and explain its physical meaning.

   (b) Provide a physical interpretation of the two branches of excitations: (i) identify the Goldstone branch corresponding to the spontaneously broken $U(1)$ symmetry and discuss the qualitative shape of its dispersion in the low-k region; (ii) compare the other mode with a Higgs-like amplitude mode of the order parameter.

   (c) Discuss the behavior of the intensity relaxation rate as a function of P_0 when the threshold is approached from above, $P_0 \to \gamma^+$. How would these intensity fluctuations appear in a photon correlation measurement?

   (d) Compare the predicted spectrum with the Bogoliubov theory of the collective excitations in a dilute Bose-Einstein condensate.