Bogoliubov theory of small fluctuations:

→ quantized Hamiltonian $H = E_0 + \sum_n \hbar \varepsilon_n^{Bog} b_n^+ b_n$

not possible because of pump / losses

→ more complex formalism of master equation

Not completely developed yet.

→ mean field equations + linearization around stationary solution

Homogeneous case:

$$i \frac{\partial}{\partial t} \psi = -\frac{\hbar \nabla^2}{2m} \psi + g|\psi|^2 \psi + i\left(P_0 - \gamma - P_1 |\psi|^2\right) \psi + \omega_0 \psi$$

below threshold: equilibrium $\psi = 0$

$$i \frac{\partial}{\partial t} \delta\psi = -\frac{\hbar \nabla^2}{2m} \delta\psi + \cancel{0} + i(P_0 - \gamma)\, \delta\psi + \omega_0 \psi$$

$$\omega\, \delta\psi_k = \frac{\hbar k^2}{2m} \delta\psi_k + i(P_0 - \gamma)\, \delta\psi_k + \omega_0\, \delta\psi_k$$

$$\omega = \omega_0 + \frac{\hbar k^2}{2m} + i(P_0 - \gamma)$$

loss rate: tends to 0 as $P_0 \to \gamma^-$

free-particle dispersion

critical slowing down

above threshold : $|\bar\psi|^2 = \frac{P_0 - \gamma}{P_1}$

$$\bar\omega = \omega_0 + g|\bar\psi|^2$$

$$\psi(x,t) = e^{-i\bar\omega t}\left[\bar\psi + \delta\psi(x,t)\right]$$

$$i\frac{\partial}{\partial t}\delta\psi + \cancel{\bar\omega\,\delta\psi} = -\frac{\hbar\nabla^2}{2m}\delta\psi + \cancel{g|\bar\psi|^2\delta\psi} + i\,(\cancel{P_0 - \gamma - P_1|\bar\psi|^2})^{0}\,\delta\psi + \cancel{\omega_0\,\delta\psi} +$$

$$+ g|\bar\psi|^2\delta\psi + g\bar\psi^2\delta\bar\psi - iP_1|\bar\psi|^2\delta\psi - iP_1\bar\psi^2\delta\psi^*$$

$$i\frac{\partial}{\partial t}\delta\psi = -\frac{\hbar\nabla^2}{2m}\delta\psi + (g - iP_1)\left(|\bar\psi|^2\delta\psi + \bar\psi^2\delta\psi^*\right)$$

assume for simplicity $\bar\psi \in \mathbb{R}$, $\bar n = |\bar\psi|^2$

$$i\frac{\partial}{\partial t}\begin{pmatrix}\delta\psi\\ \delta\psi^*\end{pmatrix} = \begin{pmatrix} -\frac{\hbar\nabla^2}{2m} + (g - iP_1)\bar n & , & (g - iP_1)\bar n \\ (-g - iP_1)\bar n & , & +\frac{\hbar\nabla^2}{2m} + (-g - iP_1)\bar n \end{pmatrix}\begin{pmatrix}\delta\psi\\ \delta\psi^*\end{pmatrix}$$

Plane-wave solution at k: $-\frac{\hbar\nabla^2}{2m} \longrightarrow \frac{\hbar k^2}{2m}$

$$\left(\frac{k^2}{2m} + (g - iP_1)\bar n - \lambda\right)\left(-\frac{k^2}{2m} - (g + iP_1)\bar n - \lambda\right) +$$

$$- (g - iP_1)\bar n\,(-g - iP_1)\bar n = 0$$

$$\left(\lambda - \left(\frac{k^2}{2m} + g\bar n\right) + iP_1\bar n\right)\left(\lambda + \left(\frac{k^2}{2m} + g\bar n\right) + iP_1\bar n\right) + (g\bar n)^2 + P_1^2\bar n^2 = 0$$

$$\left((\lambda + iP_1\bar n) - \left(\frac{k^2}{2m} + g\bar n\right)\right)\left(\lambda + iP_1\bar n) + \left(\frac{k^2}{2m} + g\bar n\right)\right) + (g\bar n)^2 + P_1^2\bar n^2 = 0$$

$$(\lambda + i P_1 \bar{n})^2 - \left(\frac{k^2}{2m} + g\bar{n}\right)^2 + g^2\bar{n}^2 + P_1^2\bar{n}^2$$

$$(\lambda + i P_1 \bar{n})^2 = \mathcal{E}_{Bog}(k)^2 - P_1^2 \bar{n}^2$$

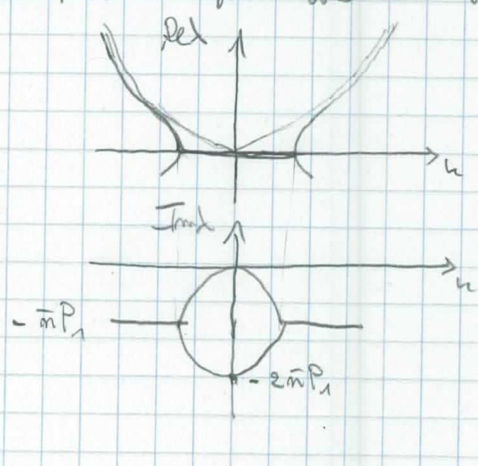
$$\boxed{\lambda = -iP_1\bar{n} \pm \sqrt{\mathcal{E}_{Bog}(k)^2 - P_1^2\bar{n}^2}}$$

* satisfies Goldstone theorem $\lambda(k\to 0) = -iP_1\bar{n} \pm \sqrt{-P_1^2\bar{n}^2}$
$= 0, -2iP_1\bar{n}$

* dynamical stability : $\exp(-i\lambda t) \simeq \exp(-i(-iP_1\bar{n})t)$
$= \exp(-P_1\bar{n}t)$

* $P_1\bar{n} = P_0 - \gamma$. Tends to 0 as $P_0 \to \gamma^+$
critical slowing down.

* Diffusive dispersion for low k such that $\mathcal{E}_{Bog}(k) < P_1\bar{n}$.



[Wouters and IC, PRL 99 140402 ('07)]

* Superfluid or not ?

Landau $\longrightarrow$ $v_c = \min_k \dfrac{\mathrm{Re}[\omega(k)]}{k} = 0$

numerically $\longrightarrow$ threshold-like behaviour of drag force $F(v)$

$\hookrightarrow$ crucial role of $\mathrm{Im}[\omega(k)]$ [Wouters, IC. arXiv: 0707.1446]

Eigenvectors:

$$\left(\frac{\hbar k^2}{2m} + (g - iP_1)\bar{n} - \lambda\right) U_k + (g - iP_1)\bar{n}\, V_k = 0$$

$$\left(\frac{\hbar k^2}{2m} + (g - iP_1)\bar{n} + iP_1\bar{n} \mp \sqrt{\varepsilon_{Bog}(k)^2 - P_1^2\bar{n}^2}\right) U_k + (g - iP_1)\bar{n}\, V_k = 0$$

$$\left(\frac{\hbar k^2}{2m} + g\bar{n} \mp \sqrt{\varepsilon_{Bog}(k)^2 - P_1^2\bar{n}^2}\right) U_k + (g - iP_1)\bar{n}\, V_k = 0$$

i) $\varepsilon_{Bog}(k) \ll P_1\bar{n}$

$$(g\bar{n} \mp iP_1\bar{n})\, U_k + (g - iP_1)\bar{n}\, V_k = 0$$

(−) Goldstone mode : $(g - iP_1)\bar{n}\, U_k + (g - iP_1)\bar{n}\, V_k = 0$

$$\Longrightarrow \quad U_k + V_k = 0$$

$$\delta\psi(x,t) = U_k e^{ikx} e^{-i\omega t} b_k + V_k^* e^{-ikx} e^{i\omega t} b_k^\dagger$$

$$\delta\rho = \underbrace{\bar{\psi}^*}_{\in \mathbb{R}}\delta\psi + \bar{\psi}\,\delta\psi^* = \bar{\psi}\,(U_k + V_k)\left(e^{ikx} e^{-i\omega t} b_k + h.c.\right)$$

$$= 0$$

→ no density modulation

→ Goldstone mode corresponds to phase rotation.

(+) mode, damped $-2iP_1\bar{n}$

$$(g + iP_1)\cancel{\bar{n}}\, U_k + (g - iP_1)\cancel{\bar{n}}\, V_k = 0$$

$$g(U_k + V_k) + iP_1(U_k - V_k) = 0$$

$$U_k + V_k = -\frac{iP_1}{g}\frac{1}{U_k + V_k}$$

$$U_k + V_k = \sqrt{-i \frac{P_1}{g}} \quad , \quad U_k - V_k = \sqrt{\frac{ig}{P_1}}$$

* for $g \to 0$: $U_k - V_k \to 0$ purely density mode

* finite g : interactions couple density and phase

Open question : role of fluctuations in destabilizing BEC

equilibrium : $H_{Bg} = \sum_k \omega_{Bg}(k)\, b_k^+ b_k$

$\to$ ground and thermal states determined by $\exp(-\beta H_{Bg})$

$\to$ non-trivial zero-point physics from Bogoliubov transformation $\delta\psi = \dots b_k + \dots b_k^+$

$\to$ quantum (and thermal) depletion of BEC

non-equilibrium : dynamics matters in determining excitation of Bogoliubov modes

$\to$ requires solution of master equation

$\to$ Bogoliubov approach can be recast in language of stochastic differential eqs for quasi-probability distribution, i.e. Wigner-W. Approximations needed to bring it into positive-2-order-diffusion form.