Phase Transitions in Nonlinear Optics

1 – Nonlinear cavity with intensity-dependent refractive index

Consider a simple model of single-mode nonlinear optical cavity, ruled by the following equation of motion for the classical field amplitude α:

$$i\frac{d\alpha}{dt} = \omega_0\alpha + \omega_{nl}|\alpha|^2\,\alpha - i\frac{\gamma}{2}\alpha + Fe^{-i\omega t}. \tag{1}$$

1. Using the theory of light transmission across Fabry-Perot cavities, give a physical interpretation to the different terms. In particular, discuss the physical meaning of ω_{nl} in terms of the intensity-dependent refractive index of the cavity material.
2. Give an expression for the incident and the transmitted intensities in terms of $|F|^2$ and $|\alpha|^2$
3. Switching to a slowly-varying variable rotating at the incident frequency, $\tilde{\alpha}(t) = \alpha(t)\,e^{i\omega t}$, write an algebraic equation for the steady-state amplitude α_{ss}. Derive from this an equation for the steady-state internal intensity $|\alpha_{ss}|^2$ in the form $f[|\alpha_{ss}|^2] = |F|^2$.
4. Plot I_{tr} as a function of I_{inc} in the two different regimes $\omega < \omega_0$ and $\omega \gg \omega_0$. Interpret the result in terms of optical limiting and optical bistability.
5. Determine the dynamical stability of these steady-states. In particular:
   (a) Linearize the evolution around the steady-state and, expressing $\tilde{\alpha}(t) = \alpha_{ss} + \delta\alpha(t)$, write explicit motion equations for the slowly varying complex variables $\delta\alpha(t)$ and $\delta\alpha^*(t)$ considered as small-valued independent variables.
   (b) Express the condition for the dynamical stability of the steady-state in terms of the eigenvalues of the linear evolution problem for $\delta\alpha$ and $\delta\alpha^*$.
   (c) In the small γ limit, relate the dynamical stability condition to the derivative $f' = d|F|^2/d|\alpha_{ss}|^2$ and, then, to the physically more transparent quantity dI_{tr}/dI_{inc}. Discuss the meaning of this condition in terms of the I_{tr} vs. I_{inc} diagrams plotted before.
   (d) Give an interpretation of these results in terms of phase transitions. How do you expect that the result is modified when quantum and classical fluctuations are included in the model (e.g. as an additive noise term in the equation of motion) ? What is the order of the phase transition?
6. Make predictions for the behaviour of a spatially extended version of this model, where the mode amplitude α is replaced by a continuous field $\alpha(\mathbf{r})$ that depends on the spatial position $\mathbf{r}$ and the pump is taken as spatially homogeneous. How do you expect that bistable behaviour is modified in this more complex model?

2 – Degenerate parametric oscillator

Consider a simple model of degenerate parametric oscillator, ruled by the following equations of motion for the classical field amplitudes α and β respectively in the fundamental and in the downconverted modes:

$$i\frac{d\alpha}{dt} = \omega_a \alpha + g\beta^2 - i\frac{\gamma_a}{2}\alpha + Fe^{-i\omega t} \quad (2)$$

$$i\frac{d\beta}{dt} = \omega_b \beta + 2g\alpha\beta^* - i\frac{\gamma_b}{2}\beta. \quad (3)$$

7. Show how the coupling terms proportional to g can be derived by an interaction Hamiltonian of the form $H_{int} = g[\hat{a}^\dagger \hat{b}^2 + (\hat{b}^\dagger)^2 \hat{a}]$. Propose a physical interpretation for the underlying microscopic process and relate the coefficient g to some nonlinear susceptibility of the cavity material. Explain the terminology "fundamental" and "downconverted" modes.

8. From now on assume that $\omega_a = 2\omega_b = \omega$ and $\gamma_a = \gamma_b$. Identify a suitable rotating reference frame where the natural frequencies of the modes drop from the equation of motions for the slowly varying amplitudes $\tilde{\alpha}(t)$ and $\tilde{\beta}(t)$.

9. Write algebraic equations determining the steady-state amplitudes α_{ss} and β_{ss} in this rotating frame. Discuss analytically their solution and make a sketchy plot of their dependence on the pump intensity F. Is there any noticeable discontinuity in this solution?

10. Identify which symmetry of the problem is spontaneously broken by the steady-state solutions and characterize the order parameter and the order of the phase transition. How is this physics analogous/different from laser oscillation? how is it analogous/different from the ferromagnetic transition in solids?

11. Characterize the phase transition in terms of the dynamical stability of the trivial solution with $\beta = 0$.

12. Make predictions for the behaviour of a spatially extended version of this model, where the mode amplitudes α and β are replaced by continuous fields $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$ that depend on the spatial position $\mathbf{r}$. Discuss the phenomenology of the phase transition as a function of the different parameters in the case of a spatially homogeneous pump.