A microscopic model for photo-detection

Consider a lossless single-mode optical cavity initially prepared in a given quantum state described by a density matrix ρ_0. The cavity contains a photodetector device weakly coupled to the cavity mode.

The probability of detecting a photon after a given interval of time is p, which is assumed to be small enough for the probability of multiple clicks to be negligible. The overall probability of clicking is then $P = p\,\mathrm{Tr}[\hat{n}\,\rho_0]$. After a click event, the quantum state of the cavity can be described by the application of a photon destruction operator,

$$\rho_{\text{click}} = \frac{\hat{a}\rho_0\hat{a}^\dagger}{\mathrm{Tr}[\hat{n}\,\rho_0]}\,. \tag{1}$$

1. How many photons are present in average in the cavity after a click event if the initial state was a Fock state with N photons, $\rho_0 = |N\rangle\langle N|$?
2. How many photons are present in average in the cavity if the initial state was a coherent state of amplitude α, namely $\rho_0 = |\mathrm{coh}:\alpha\rangle\langle\mathrm{coh}:\alpha|$?
3. How many photons are present in average in the cavity if the cavity was initially in a thermal state $\rho_0 = Z^{-1}\,e^{-\beta\hbar\omega\,\hat{a}^\dagger\hat{a}}$ with average photon number equal to N ?
4. Explain these results in terms of the second order intensity correlation function $g^{(2)}$ of the cavity field and of the photon number distribution after the first detection event.

In an alternative picture, the photo-detection process can be microscopically modeled as a short unitary evolution under a beam-splitter Hamiltonian mixing the cavity mode $\hat{a}$ with an initially empty auxiliary mode $\hat{b}$,

$$\hat{U} = e^{-i\varepsilon[\hat{a}^\dagger\hat{b}+\hat{b}^\dagger\hat{a}]}\,, \tag{2}$$

followed by a measurement of the photon number in the $\hat{b}$ mode and a reinitialization of this mode into its vacuum state.

5. Within the Heisenberg picture, show that the field operators $\hat{a}_{aft}$ and $\hat{b}_{aft}$ after the action of U are related to the ones before U by:

$$\begin{aligned} \hat{a}_{aft} &= \cos(\varepsilon)\,\hat{a}_{bef} - i\sin(\varepsilon)\,\hat{b}_{bef} \qquad (3) \\ \hat{b}_{aft} &= \cos(\varepsilon)\,\hat{b}_{bef} - i\sin(\varepsilon)\,\hat{a}_{bef}\,. \qquad (4) \end{aligned}$$

   To this purpose, it can be useful to first find a Hamiltonian that recovers the evolution $\hat{U}$ and then write the corresponding Heisenberg equations of motion for $\hat{a}$ and $\hat{b}$. Relate the mixing angle ε to the detection probability p considered above.

6. Write the explicit expression of the density matrix ρ_0 of the three states considered above in terms of creation operators $\hat{a}^\dagger$ acting on the vacuum state. Use this expression to write the density matrix after the detection process ρ_{aft} in terms of the Heisenberg picture operators $\hat{a}_{aft}$ and, using the explicit expressions Eqs.(3-4), in terms of the Schrödinger-picture creation operators $\hat{a}^\dagger$ and $\hat{b}^\dagger$ acting on vacuum.

7. In the small p limit, project the density matrix ρ_{aft} on the one-photon state of the b mode. Show that, for the three initial (Fock, coherent, thermal) states considered above, this projected density matrix $\rho^{(1)}_{aft}$ matches the one predicted by Eq.(1) for the same case and well describes the state after a click event.

8. Show that, for small p, the probability of multiple clicks, namely of having more than one photon in the b mode is indeed negligible.