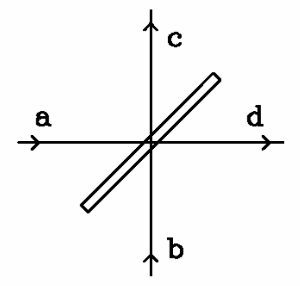
Quantum theory of the beam splitter



Consider the model of beam-splitter that is sketched in the figure. Light is incident from the a, b input arms and is transmitted/reflected into the c, d output arms. In the classical theory, the beam-splitter operation is summarized by the scattering matrix $\mathbf{S}(\omega)$ connecting the output fields to the input ones at a given frequency ω:

$$\begin{pmatrix} E_c(\omega) \\ E_d(\omega) \end{pmatrix} = \mathbf{S}(\omega) \begin{pmatrix} E_a(\omega) \\ E_b(\omega) \end{pmatrix}. \tag{1}$$

For a lossless beam-splitter, energy conservation arguments allow to write $\mathbf{S}$ in the canonical form:

$$\mathbf{S}(\omega) = \begin{pmatrix} t(\omega) & ir(\omega) \\ ir(\omega) & t(\omega) \end{pmatrix} \tag{2}$$

with r, t real functions of ω and $t(\omega)^2 + r(\omega)^2 = 1$.

In a quantum description of light, the matrix $\mathbf{S}$ has the meaning of scattering S-matrix connecting the amplitude of the single-photon eigenfunction of energy $\omega = ck$ on the c, d arms to the amplitude of the same wavefunction on the a, b arms:

$$\begin{pmatrix} \bar{\psi}_{c,\omega} \\ \bar{\psi}_{d,\omega} \end{pmatrix} = \mathbf{S}(\omega) \begin{pmatrix} \bar{\psi}_{a,\omega} \\ \bar{\psi}_{b,\omega} \end{pmatrix}. \tag{3}$$

As usual, the eigenfunction has a plane wave spatial profile on each arm, e.g.

$$\psi_{a,\omega}(x) = e^{ikx}\, \bar{\psi}_{a,\omega}. \tag{4}$$

1. The states of the photon in the a, b, c, d arms at a wavevector k can be labelled as $|(a, b, c, d); k\rangle$. The $\hat{\mathbf{U}}$ operator describing the evolution of the photon from an early time t_i to a late time t_f sends the two-dimensional $|(a, b); k\rangle$ space into the $|(c, d); k\rangle$ space. Show that the matrix elements of the $\hat{\mathbf{U}}$ operator in this basis correspond to the elements of the $\mathbf{S}$ matrix described above multiplied by an overall phase factor $\exp[-ick(t_f - t_i)]$ due to the photon frequency.

2. Show that the action of $\hat{\mathbf{U}}$ on the creation operators $\hat{a}_k^\dagger$, $\hat{b}_k^\dagger$ for the $|(a,b);k\rangle$ states is:

$$\begin{aligned} \hat{\mathbf{U}}\,\hat{a}_k^\dagger\,\hat{\mathbf{U}}^\dagger &= t(k)\,\hat{c}_k^\dagger + ir(k)\,\hat{d}_k^\dagger && (5)\\ \hat{\mathbf{U}}\,\hat{b}_k^\dagger\,\hat{\mathbf{U}}^\dagger &= ir(k)\,\hat{c}_k^\dagger + t(k)\,\hat{d}_k^\dagger, && (6)\end{aligned}$$

   where $\hat{c}_k^\dagger$, $\hat{d}_k^\dagger$ are the creation operators for the $|(c,d);k\rangle$ states.

3. Write the corresponding equations for the destruction operators.

4. Write the late-time out-going destruction operators $\hat{\mathbf{U}}^\dagger\hat{c}_k\hat{\mathbf{U}}$, $\hat{\mathbf{U}}^\dagger\hat{d}_k\hat{\mathbf{U}}$ in terms of the early-time in-going ones $\hat{a}_k$, $\hat{b}_k$. Interpret these expression in the Heisenberg picture of quantum mechanics.

5. Show that a coherent state is obtained in the output for any generic input coherent state (neglect for simplicity the k dependence):

$$|\psi\rangle = \exp\left\{-[|\alpha|^2 + |\beta|^2]/2\right\}\,\exp(\alpha\hat{a}^\dagger + \beta\hat{b}^\dagger)|\mathrm{vac}\rangle. \qquad (7)$$

   Show that the outgoing coherent state fulfills the classical relations for the reflection/transmission amplitudes discussed above.

6. What state is obtained in the output if two photons are simultaneously incident on a perfectly 50/50 beam-splitter, one from arm a and one from arm b ? Can this effect be described in a classical picture ?

7. (Optional) What happens if the two photons incide on the beam-splitter at different times? For this question, one has to explicitely take into account the k dependence of field operators, etc.