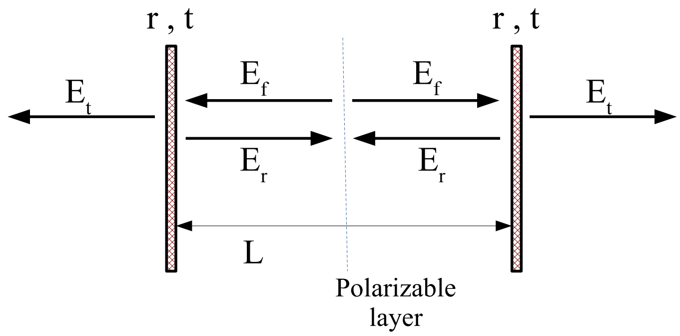
Coupled-mode theory and strong light-matter coupling

Consider a one-dimensional model of a two-sided cavity of length L enclosed by a pair of lossless mirrors of transmittivity t and reflectivity r, with real $r, t > 0$. A polarizable layer is located at the center of the cavity and is coupled to the cavity field.



In this first part of the exercise, consider the current in the polarizable layer to be externally determined and have a complex-valued amplitude J_0 and a frequency ω_0 close to a given resonance of the cavity.

1. Starting from the general expression for the field emitted by the cavity

$$E_t = -\frac{2\pi t J_0}{c}\frac{1}{1 - re^{i\omega_0 L/c}} \tag{1}$$

   find an approximate expression valid in the case where the dipole frequency ω_0 is close to a given cavity resonance at ω_c and the cavity has a high finesse.

2. Express the possible values of the resonance frequency ω_c in terms of the cavity parameters. Draw the spatial profile of the field of the different cavity modes.

3. Making use of the relation $E_t = tE_f$ relating the emitted field to the in-cavity field and of the $\omega \to i\partial_t$ relation between frequency- and time-domains, write a differential equation for the dynamics of the cavity field amplitude.

4. Interpret the current as a driving force acting on a damped harmonic oscillator.

In this second part, the evolution of the current in the polarizable layer is self-consistently determined by its interaction with the cavity field. To this purpose, consider a Drude-Lorentz-like model for the dynamics of the polarization d, assuming a resonance frequency ω_0, effective mass m and charge q, and an effective density n_{2d} along the layer.

5. Write an expression for the field amplitude at the location of the polarizable layer in terms of the cavity field E_f and the reflected field $E_r = rE_f$.
6. Write an equation of motion for the polarization d. Find an approximated form valid in the vicinity of the resonance that only involves a first temporal derivative. Interpret the role of the cavity field as a driving force acting on the polarization.
7. Approximating the current as $J \simeq -i\omega_0 d$, write a full system dynamics as a system of two coupled first-order differential equations for the polarization d and the field amplitude E_f. This is the so-called *coupled mode theory.*
8. Find the eigenmodes of the system and their eigenfrequencies. Express the minimum value of the splitting between the eigenmodes, the so-called *Rabi splitting*, in terms of the system parameters.
9. Interpret the eigenmodes as polaritonic resonances and compare them with the polariton modes of a cavity filled with a Drude-Lorentz medium. Provide an interpretation for the relative value of the Rabi frequency in the two cases.