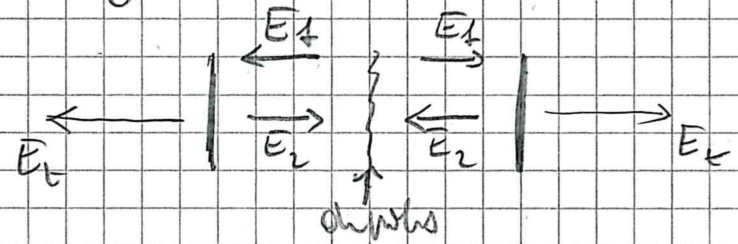
Exercise 15 : Coupled-mode theory and strong light-matter coupling

Cavity as at 3.10 and following



$$E_1 = -\frac{2\pi J_0}{c}\,\frac{1}{1-r e^{i\omega L/c}} \simeq -\frac{2\pi i J_0}{L}\,\frac{1}{\omega-\omega_c+i\Gamma/2}$$

(holds for $|\omega-\omega_c| \ll \omega_{FSR}$) (*)

$$\Gamma = cT/2L$$

Rewrite in t-domain via $\omega \to i\frac{d}{dt}$

$$i\frac{dE_1}{dt} = \omega_c E_1 - i\frac{\Gamma}{2}E_1 - \frac{2\pi i}{L}J_0$$

Dipoles : $m\ddot{x} = -m\omega_0^2 x + qE$

↳ located at center $E = E_1 + E_2 \simeq 2E_1$

$$m(\omega_0^2-\omega^2)\,x = qE$$

RWA condition $|\omega-\omega_0| \ll \omega_0$

$$2m\omega_0(\omega_0-\omega) = qE$$

$$i\frac{dx}{dt} = \omega_0 x - \frac{q}{2m\omega_0}E$$

Coupled light-matter dynamics :

$$J_0 = \sigma_{2D}\, q\, \dot{x} = -i\omega\, \sigma_{2D}\, q\, x$$

$$i \frac{d}{dt}\begin{pmatrix} E_1 \\ x \end{pmatrix} = \begin{pmatrix} \omega_c - i\Gamma/2 & -\frac{2\pi\omega\,\sigma_{2D}\, q}{L} \\ -\frac{q}{2m\omega_0}\cdot 2 & \omega_0 \end{pmatrix} \begin{pmatrix} E_1 \\ x \end{pmatrix}$$

$\hookrightarrow$ eigenvalues give eigenmodes

For simplicity $\Gamma = 0$:

$$\omega_\pm = \frac{1}{2}(\omega_c + \omega_0) \pm \sqrt{\left(\frac{\omega_c - \omega_0}{2}\right)^2 + \Omega^2}$$

"Rabi coupling" $\Omega = \sqrt{\left(-\frac{q}{\cancel{2}m\cancel{\omega_0}}\cdot\cancel{2}\right)\left(-\frac{2\pi\cancel{\omega}\,\sigma_{2D}\, q}{L}\right)}$

$$\simeq \sqrt{\frac{2\pi q^2}{mL}\sigma_{2D}}$$

Remind : eq. (*) requires $\Omega \ll \omega_{FSR} = \frac{\pi c}{L}$

Filled cavity: from 1.8 we have

$$\frac{\omega_0^2}{c^2} = k^2 = \left(\frac{\pi}{L}\right)^2 = \frac{\omega^2}{c^2}\left[1 + \frac{4\pi q^2 \sigma_{3d}}{m} \frac{1}{\omega_0^2 - \omega^2}\right]$$

fix cavity length to resonance $\omega_c = \omega_0$

Eigenmodes @ $\omega_0 + \delta\omega$ imposes:

$$\cancel{\frac{\omega_0^2}{c^2}} \simeq \frac{\omega_0^2 + 2\omega_0 \delta\omega}{c^2}\left[1 + \frac{4\pi q^2 \sigma_{3d}}{m} \frac{1}{-2\omega_0 \delta\omega}\right]$$

$$\simeq \cancel{\frac{\omega_0^2}{c^2}} + \frac{2\omega_0}{c^2}\delta\omega - \frac{\omega_0}{2c^2} \frac{4\pi q^2 \sigma_{3d}}{m} \frac{1}{\delta\omega}$$

+ small terms

$$\Rightarrow \delta\omega = \pm\left(\frac{\pi q^2 \sigma_{3d}}{m}\right)^{1/2}$$

Same formula if we set $\sigma_{3d} = \frac{\sigma_{2d}}{L}$ except for factor $\sqrt{2}$ which accounts for better field-emitter overlap for localized emitters