Second Harmonic Generation

Consider a planar ensemble of two-dimensional density n_{2d} of charged harmonic oscillators. Each oscillator has mass m, charge q and bare frequency ω_0, is forced by an incident monochromatic field $E_1(t)$ at frequency ω, and is subject to a weak non-linear restoring force, as described by the motion equation

$$m\ddot{x} = -m\omega_0^2 x - m\gamma\dot{x} + qE_1(t) - \beta x^2\,. \tag{1}$$

Here, β is the strength of the nonlinear coupling, the driving force is $E_1(t) = \bar{E}_1 e^{-i\omega t} + \text{c.c.}$, and $\gamma = 2\pi n_{2d} q^2/mc$ is the radiative damping of the oscillator due to light emission.

1. Derive the form of the anharmonic potential leading to the nonlinear force $-\beta x^2$.
2. At zeroth order in β, derive the amplitude $A_1^{(0)}$ of the steady-state oscillation at ω.
3. Estimate the time-dependent force that is generated by the nonlinear coupling term under the zeroth order oscillation.
4. Calculate the first order correction to the steady-state oscillation due to the nonlinear force. Show that this involves a static displacement and an oscillation at frequency 2ω. Estimate the amplitude $A_2^{(1)}$ of this second harmonic oscillation. Interpret the static term in terms of optical rectification in the anharmonic potential.
5. Evaluate the emitted power at the second harmonic frequency 2ω as a function of $A_1^{(0)}$.
6. Going to second order in β, evaluate the component of the nonlinear force that oscillates at the fundamental frequency ω as a function of $A_1^{(0)}$.
7. Evaluate the work done by this nonlinear force as a function of $A_1^{(0)}$ and give a physical interpretation of the result by connecting this work to the radiated second harmonic power.