Exercise 13 : Second Harmonic Generation

1) $V = -\int_0^x F\,dx = -\int_0^x -\beta x^2\,dx = \frac{\beta x^3}{3}$

2) $m\ddot{x} = -m\omega_0^2 x - m\gamma\dot{x} + q\left(\bar{E}_1 e^{-i\omega t} + \bar{E}_1^* e^{i\omega t}\right)$

$$x(t) = \underbrace{\frac{q\bar{E}_1/m}{\omega_0^2 - \omega^2 - i\gamma\omega}}_{A_1^{(0)}} e^{-i\omega t} + c.c.$$

3) $-\beta x(t)^2 = -\beta\left[A_1^{(0)} e^{-i\omega t} + A_1^{(0)*} e^{i\omega t}\right]^2 =$

$$= \underbrace{-\beta\left[A_1^{(0)2} e^{-2i\omega t} + c.c.\right.}_{\delta F_2(t)} + \underbrace{\left.2|A_1^{(0)}|^2\right]}_{\delta F_0}$$

4) By linearity of the motion equations

$$m\delta\ddot{x}_2 = -m\omega_0^2\delta x_2 - m\gamma\delta\dot{x}_2 + \delta F_2(t)$$

$$\cancel{m\delta\ddot{x}_0} = -m\omega_0^2\delta x_0 - \cancel{m\gamma\delta\dot{x}_0} + \delta F_0 \Rightarrow \delta x_0 = \frac{\delta F_0}{m\omega_0^2} =$$

$$= \frac{-2\beta|A_1^{(0)}|^2}{m\omega_0^2}$$

$$\delta x_2(t) = A_2^{(1)} e^{-2i\omega t} + c.c.$$

$$m\left(\omega_0^2 - (2\omega)^2 - i\gamma(2\omega)\right) A_2^{(1)} = -\beta A_1^{(0)2}$$

$$A_2^{(1)} = \frac{-\beta A_1^{(0)2}}{m\left(\omega_0^2 - (2\omega)^2 - 2i\gamma\omega\right)}$$

5) Radiative force $-m\gamma\dot{x} = F_{rad}$

$$W_{rad} = -\langle -m\gamma\dot{x}\cdot\dot{x}\rangle = m\gamma\langle \dot{x}^2\rangle$$

isolate SHG contribution (note: no interference)

$$W_{rad}^{SHG} = m\gamma \cdot 2\,\mathrm{Re}\left|-i2\omega A_2^{(1)}\right|^2 = 8m\gamma\omega^2\left|A_2^{(1)}\right|^2$$

$$= 8m\gamma\frac{\beta^2\left|A_1^{(0)}\right|^4\omega^2}{m^2\left[\left(\omega_0^2-(2\omega)^2\right)^2+4\gamma^2\omega^2\right]} =$$

$$= \frac{8\gamma\omega^2\beta^2}{m}\frac{\left|A_1^{(0)}\right|^4}{\left(\omega_0^2-(2\omega)^2\right)^2+4\gamma^2\omega^2}$$

6) Among all terms of $-\beta x^2$ at 1^{st} order in β

$$\delta F_1(t) = -2\beta A_2^{(1)} A_1^{(0)*} e^{-i\omega t} + c.c. =$$

$$= \frac{2\beta^2\left|A_1^{(0)}\right|^2 A_1^{(0)}}{m\left(\omega_0^2-(2\omega)^2-2i\gamma\omega\right)}$$ is only component at ω

$$W = 2\,\mathrm{Re}\left(\delta F_1 \,(\dot{X})^*\right) =$$

$$= \frac{4\beta^2 |A_1^{(0)}|^4 \;\mathrm{Re}\left(+i\omega\cdot(2i\gamma\omega)\right)}{m\left(\left(\omega_0^2-(2\omega)^2\right)^2+4\gamma^2\omega^2\right)} =$$

$$= -\frac{8\beta^2\omega^2\gamma}{m\left(\left(\omega_0^2-(2\omega)^2\right)^2+4\gamma^2\omega^2\right)}\cdot |A_1^{(0)}|^4$$ work done onto oscillation at ω

which matches the power emitted by SHG

$\Rightarrow$ Effective nonlinear radiative friction

$$2m\,\gamma_{SHG}\left(\omega^2 |A_1^{(0)}|^2\right) = W$$

$$\gamma_{SHG} = \frac{4\beta^2}{m^2}\,\frac{\gamma}{\left(\omega_0^2-(2\omega)^2\right)^2+4\gamma^2\omega^2}\,|A_1^{(0)}|^2$$

felt by oscillation at ω