A classical model for superradiance

Consider a forced harmonic oscillator with the usual motion equation

$$m\ddot{x} = -\kappa x - m\gamma\dot{x} + qE(t) \tag{1}$$

but a negative mass $m < 0$ and negative spring constant $\kappa < 0$. The damping constant is instead positive $\gamma > 0$ as usual.

1. Show that the conservative part of the equation of motion (1) can be derived from the usual Hamiltonian,
$$H = \frac{p^2}{2m} + \frac{m\omega_0^2}{2}x^2 - qE(t)x. \tag{2}$$
Interpret the $m, \kappa < 0$ condition in terms of the sign of the Hamiltonian.

2. Describe the dynamics of this "inverted" harmonic oscillator in the absence of driving field $E = 0$. Is the stationary state at $x = 0$ *dynamically* stable?

3. Discuss briefly the thermodynamical equilibrium state of the system in the absence of driving field $E = 0$ and at a finite temperature T in terms of the Canonical Ensemble in the phase space. Is the system *thermodynamically* stable at $T > 0$?

4. Write the differential equation for the time-evolution of the kinetic energy $m\dot{x}^2/2$ and identify the term due to the damping. Verify whether the $\gamma > 0$ choice leads to a physically meaningful model of mechanical friction and discuss the limits of validity of this model.

5. Suggest a physical realization of the inverted harmonic oscillator in a solid-state context. Work out a physical connection to the dynamics of the dipole moment of a population-inverted atom.

Include now a one-dimensional electromagnetic field incident from $z < 0$ onto a sheet of two-dimensional density σ of such dipoles located at $z = 0$. Study the field that is scattered from the current that the incident field induces in the dipole sheet. Indicating by q the charge of each dipole, the reflection and transmission amplitudes read:

$$r(\omega) = \frac{E_r}{E_{inc}} = \frac{\frac{2\pi i\sigma q^2\omega}{mc}}{\omega_0^2 - \omega^2 - i\gamma\omega - \frac{2\pi i\sigma q^2\omega}{mc}} \tag{3}$$

$$t(\omega) = \frac{E_t}{E_{inc}} = \frac{\omega_0^2 - \omega^2 - i\gamma\omega}{\omega_0^2 - \omega^2 - i\gamma\omega - \frac{2\pi i\sigma q^2\omega}{mc}} \tag{4}$$

6. Verify that these formulas derived in Lecture 3 for standard harmonic oscillators ($m > 0$, $\kappa > 0$) also hold in the inverted $m, \kappa < 0$ case with the usual definition of a real-valued resonance frequency $\omega_0 = \sqrt{\kappa/m}$.

7. For small enough (a precise condition will be derived in the following of the exercise) values of σ, discuss the spectral form of the reflectivity and of the transmittivity. What are the novelties of the inverted $m, \kappa < 0$ case?

8. Study in particular the resonant transmittivity for $\omega = \omega_0$ as a function of γ and of the (now negative) radiative linewidth $\gamma_{rad} = 2\pi\sigma q^2/mc$. Give a physical explanation for the obtained enhanced transmittivity in terms of energy balance.

Let's now look for solutions of the full motion equations (field and harmonic oscillator) in the absence of any incident field.

9. Explain why such solutions only exist at the poles of the transmission and reflection amplitudes $r(\omega)$ and $t(\omega)$ for complex frequencies ω. Evaluate the position $\bar{\omega}$ of the poles in the complex-ω plane and give a physical interpretation to the fact that $t(\bar{\omega}) = r(\bar{\omega})$ at the pole.

10. Discuss the spatial shape along z of generic solutions of the field equations for a general complex frequency ω in a spatially homogeneous geometry in free space with no dipoles. For the geometry under consideration in this problem, for each of the two piecewise homogeneous regions $z \gtrless 0$, identify the "out-going" solutions that do not involve any wave incident from $z = \pm\infty$ onto the dipole sheet located at $z = 0$. Explain the physical meaning of the imaginary part of the wavevector.

11. Build and characterize the full solution (field and harmonic oscillator) for $\omega = \bar{\omega}$ in the absence of incident field. Explain the different physical nature of such solutions in the two cases of $\gamma > |\gamma_{rad}|$ and of $\gamma < |\gamma_{rad}|$. Interpret the result in terms of ring-down oscillations and of dynamical instability.

At the end of the Quantum Optics course you will be able to consider the additional questions: what happens when we consider a quantum rather than classical electromagnetic field? How is the stationary state with the harmonic oscillators at rest and a vanishing electromagnetic field modified by zero-point quantum fluctuations? Is there any emission from the dipole sheet in its quantum ground state?