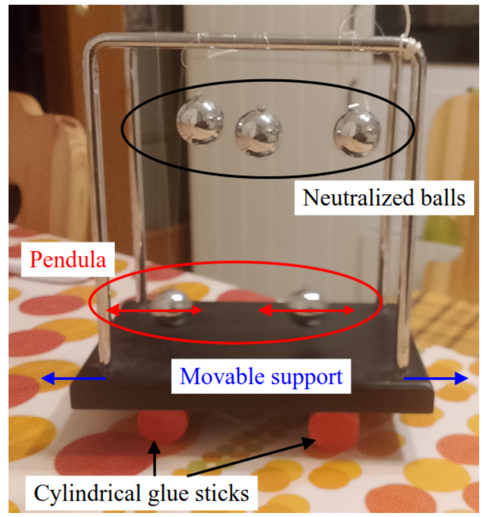
Non-Hermitian physics with pendula

The goal of this exercise is to understand the mechanism for the mechanical coupling between a pair of pendula when they are mounted on a mobile support. An example of hand-made realization of the system is shown in the Figure, namely a second-hand Newton's craddle toy where only two pendula are left free to oscillate. The whole object is put on a pair of cylindrical glue sticks and is thus free to roll back and forth, although with a significant friction. The frequency of the two pendula can be tuned by changing the length of the ropes holding the metallic balls.



Part A: Theoretical preliminaries

1. Consider a pair of harmonic oscillators described, within the Rotating Wave Approximation, by the following equations of motion

$$i\dot{\alpha}_1 = (\omega_0 + \delta/2)\alpha_1 + \Omega\alpha_2 \tag{1}$$
$$i\dot{\alpha}_2 = (\omega_0 - \delta/2)\alpha_2 + \Omega'\alpha_1 \,. \tag{2}$$

Determine the frequencies of the eigenmodes as a function of the detuning δ in the Hermitian $\Omega^* = \Omega'$ and non-Hermitian $\Omega^* = -\Omega'$ cases. Highlight the crucial differences in the behaviour in the two cases.

2. A model of the experimental system can be built starting from the Lagrangian

$$\mathcal{L} = \frac{m_1}{2}(\dot{x}_1)^2 + \frac{m_2}{2}(\dot{x}_2)^2 + \frac{M}{2}(\dot{y})^2 - \frac{\kappa}{2}(x_1 - y)^2 - \frac{\kappa}{2}(x_2 - y)^2 \tag{3}$$

where $x_{1,2}$ are the positions of the two balls and y is the position of the support. The masses $m_{1,2}$ are allowed to be slightly different, while the effective elastic constant κ is assumed for simplicity to be the same for the two pendula.

(a) Justify the chosen form of the Lagrangian. Write the Euler-Lagrange equations of motion.

(b) We can assume that the strongest friction acts on the support, with a damping term $-M\Gamma\dot{y}$ in the corresponding kotion equation. Taking Γ to be the largest frequency scale, write an equation relating the instantaneous speed $\dot{y}$ to the instantaneous position variables $x_{1,2}$ and y.

(c) Assuming that the rest of the dynamics occurs in a narrow frequency window, make use of a rotating wave approximation to eliminate the support variable y and write effective damping terms in the equation of motion for $x_{1,2}$.

(d) Moving to the symmetric/antisymmetric variables $x_{S,A} = (x_1 \pm x_2)/\sqrt{2}$, characterize the effective damping and discuss its strength as a function of the physical damping Γ.

(e) Making use of a rotating wave approximation, derive equations of motion in the form (1-2).

<u>Part B:</u> Experiments

1. Build a home-made experimental set-up using your favourite building blocks. Don't hesitate changing the elements to improve the overall performance of the set-up.

2. Put the device directly on the table with no underlying glue sticks.

   (a) Putting a single pendulum in motion, verify that a negligible energy is transferred to the other pendulum. Verify that this behaviour persists independently of the relative frequency of the pendula.

   (b) When the frequencies are (slightly) differentc and both pendula are initially excited, observe how the relative phase of the oscillation the two pendula evolves in time.

3. Put the home-made realization of the device on the glue sticks and ensure that the support is free to move.

   (a) Putting a single pendulum in motion, verify that energy gets redistributed among the two pendula.

   (b) Tuning the pendula to have approximately equal frequencies, observe the late-time behaviour of the oscillations before these are totally damped out. What is the relative phase of oscillation?

   (c) Repeat the observation when the pendula are tuned to have significantly different frequencies. Identify the main differences in the late-time behaviour.

(d) Repeat these experiment starting from an initial condition where the pendula are displaced either symmetrically or antisymmetrically. In particular, observe if there is mixing between the symmetric/antisymmetric modes and if there are substantial differences in the damping rates. Connect the result of this experiment to the previous observations.

(e) Make use of the theoretical model to explain the observations.

Part C: Going further

1. The brave experimentalists may develop an automatic acquisition set-up to extract the position of the two balls and of the support from a movie of the evolution. To this purpose, LED lights of different colors may be attached to the objects and the experiment may be performed in dark. Use this information to fit the time-evolution of the system and quantitatively measure the complex frequencies of the eigenmodes in the different cases.

2. The brave theorists may try to explain why our pendula stabilize to an opposite-phase oscillation whereas the metronomes in this movie stabilize to an in-phase oscillation.