Solution of Exercise 17: A microscopic model for photo-detection

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1 A microscopic model for photodetection

Consider a lossless single-mode optical cavity initially prepared in a given quantum state described by a density matrix ρ_0. The cavity contains a photodetector device weakly coupled to the cavity mode.

The probability of detecting a photon after a given interval of time (after turning on the photodetector) is p, which is assumed to be small enough for the probability of multiple clicks to be negligible. The overall probability of clicking is then $P = p\text{Tr}[\hat{n}\rho_0]$, where $\hat{n}$ is the number operator. After a click event, the quantum state (i.e. the new density matrix) of the cavity can be described by the application of a photon destruction operator,

$$\rho_{click} = \frac{\hat{a}\rho_0\hat{a}^\dagger}{\text{Tr}[\hat{n}\rho_0]} \tag{1}$$

1.1 Case 1: $\rho_0 = |N\rangle\langle N|$ [Fock state]

Let's consider as our initial state a Fock state with N photons. How many photons are present on average in the cavity after a click event? We can directly calculate the new density matrix using (1), remembering that $\hat{a}|N\rangle = \sqrt{N}|N-1\rangle$ and $\hat{n} = \hat{a}^\dagger\hat{a}$:

$$\rho_{click} = \frac{\hat{a}|N\rangle|N\rangle\hat{a}^\dagger}{\text{Tr}[\hat{n}|N\rangle\langle N|]} = \frac{N|N-1\rangle\langle N-1|}{N} = |N-1\rangle\langle N-1|$$

Thus, the average number of photons after a click will be simply

$$\langle n\rangle = \text{Tr}[\hat{n}\rho_{click}] = N - 1$$

We conclude that after a click event the overall probability of detecting a photon has slightly decreased (because it depends directly on the average number of photons in the cavity, since $P(t) = p(t)\langle n\rangle_t$), but this fact could be expected. In fact, if we look at the photocorrelation function for different times $g^{(2)}(t,t')$, with $t' > t$, we know that this correlation value will be less than 1 for Fock states, meaning that if we detect at time t one photon, we have a reduced probability of finding another one immediately after that measure.

1.2 Case 2: $\rho_0 = |coh:\alpha\rangle\langle coh:\alpha|$ [coherent state]

Let's consider as our initial state a coherent state with amplitude α. How many photons are present on average in the cavity after a click event? We can directly calculate the new density matrix using (1), remembering that by definition of coherent states $\hat{a}|coh:\alpha\rangle = \alpha|coh:\alpha\rangle$:

$$\rho_{click} = \frac{\hat{a}|coh:\alpha\rangle\langle coh:\alpha|\hat{a}^\dagger}{\text{Tr}[\hat{n}|coh:\alpha\rangle\langle coh:\alpha|]} = \frac{|\alpha|^2|coh:\alpha\rangle\langle coh:\alpha|}{|\alpha|^2} = |coh:\alpha\rangle\langle coh:\alpha|$$

Thus, the average number of photons after a click will be simply

$$\langle n\rangle = \text{Tr}[\hat{n}\rho_{click}] = |\alpha|^2$$

We conclude that after a click event the overall probability of detecting a photon has remained the exactly thr same, but this fact could be expected. In fact, if we look at the photocorrelation function for different times $g^{(2)}(t,t')$, with $t' > t$, we know that this correlation value will be equal to 1 for coherent states, meaning that if we detect at time t one photon, we have a the same probability as before of finding another one immediately after that measure.

1.3 Case 3: ρ_0 as thermal state, with $\langle n_0\rangle = N$

Let's consider as our initial state a thermal state $\rho_0 = Z^{-1}e^{-\beta\hbar\omega\hat{a}\hat{a}^\dagger}$ with average photon number equal to N. How many photons are present on average in the cavity after a click event? We can directly calculate the new density matrix using (1), remembering the cited above properties of the ladder operators when applied to Fock states:

$$\rho_{click} = \frac{1}{Z} \frac{\hat{a} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} |n\rangle \langle n| \hat{a}^\dagger}{\mathrm{Tr}[\hat{n}\rho_0]}$$

Note that the average number of photons calculated on ρ_0 will be equal to N by our initial assumption, but also to the specific Bose population factor for a certain temperature β:

$$\mathrm{Tr}[\hat{n}\rho_0] = \frac{1}{Z} \sum_{n=0}^{\infty} n e^{-\beta\hbar\omega n} = \frac{1}{e^{\beta\hbar\omega} - 1} = N$$

Moreover, also the partition function can be recast in terms of N multiplied by an exponential, as follows:

$$Z = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \frac{1}{1 - e^{-\beta\hbar\omega}} = \frac{e^{\beta\hbar\omega}}{e^{\beta\hbar\omega} - 1} = N e^{\beta\hbar\omega}$$

We insert these new expressions inside ρ_{click} and we end up with this expression:

$$\rho_{click} = \frac{e^{-\beta\hbar\omega}}{N^2} \sum_{n=1}^{\infty} n e^{-\beta\hbar\omega n} |n-1\rangle \langle n-1|$$

At this point, we can evaluate what will be the average number of photons after a click:

$$\begin{aligned}
\langle n \rangle = \mathrm{Tr}[\hat{n}\rho_{click}] &= \frac{e^{-\beta\hbar\omega}}{N^2} \mathrm{Tr}\left[\sum_{n'=0}^{\infty} \sum_{n=1}^{\infty} n n' e^{-\beta\hbar\omega n} |n-1\rangle \langle n-1| |n'\rangle \langle n'| \right] = \\
&= \frac{e^{-\beta\hbar\omega}}{N^2} \mathrm{Tr}\left[\sum_{n=2}^{\infty} n(n-1) e^{-\beta\hbar\omega n} |n-1\rangle \langle n-1| \right] = \\
&= \frac{e^{-\beta\hbar\omega}}{N^2} \sum_{n=2}^{\infty} n(n-1) e^{-\beta\hbar\omega n} = \\
&= \frac{e^{-3\beta\hbar\omega}}{N^2} \frac{d^2}{dz^2} \left[\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} \right] = \text{(derivative of the geometric series)} \\
&= \frac{e^{-3\beta\hbar\omega}}{N^2} \frac{2}{(1 - e^{-\beta\hbar\omega})^3} = \frac{2N^3}{N^2} = 2N
\end{aligned}$$

We conclude that after a click event the overall probability of detecting a photon has been increased by a factor 2, but this fact could be expected. In fact, if we look at the photocorrelation function for different times $g^{(2)}(t,t')$, with $t' > t$, we know that this correlation value will be almost equal to 2 for thermal states when $|t-t'| \ll 1$, meaning that if we detect at time t one photon, we have a *super-poissonic* probability of finding another one immediately after that measure. In other words, photons are said to arrive in bunches.

2 An alternative picture: the mixing Hamiltonian

In an alternative picture, the photo-detection process can be microscopically modeled as a short unitary evolution under a beam-splitter Hamiltonian mixing the cavity mode $\hat{a}$ with an initially empty auxiliary mode $\hat{b}$,

$$\hat{U} = e^{-i\epsilon[\hat{a}^\dagger\hat{b}+\hat{b}^\dagger\hat{a}]} \tag{2}$$

followed by a measurement of the photon number in the $\hat{b}$ mode and a reinitialization of this mode into its vacuum state. Please notice that from now on we will refer to ϵ as a sort of mixing angle, but in reality this should be an infinitesimal amount of time during which we let the cavity interact with the photodetector, and collecting some photons on it.

Within the Heisenberg picture, we are interested in finding the time evolution of the operators $\hat{a}$ and $\hat{b}$ after the short mixing interaction. Looking at the short unitary propagator $\hat{U}$, we define the interaction Hamiltonian H_{int} as

$$H_{int} = \hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}$$

Recalling the commutation relations $[\hat{a},\hat{a}^\dagger] = [\hat{b},\hat{b}^\dagger] = 1$, we write the 2 Heisenberg equations as (for simplicity, we consider $\hbar = 1$)

$$\begin{aligned}\frac{\partial\hat{a}}{\partial t} &= \frac{1}{i}[\hat{a},H_{int}] = \frac{1}{i}[\hat{a},\hat{a}^\dagger\hat{b}+\hat{b}^\dagger\hat{a}] = -i\hat{b}\\ \frac{\partial\hat{b}}{\partial t} &= \frac{1}{i}[\hat{b},H_{int}] = \frac{1}{i}[\hat{b},\hat{a}^\dagger\hat{b}+\hat{b}^\dagger\hat{a}] = -i\hat{a}\end{aligned}$$

This is a system of coupled differential equations, which can be easily decoupled by taking the second derivative of one equation and inserting the other one inside:

$$\frac{\partial^2 \hat{a}}{\partial t^2} = -i\frac{\partial \hat{b}}{\partial t} = -\hat{a} \quad \rightarrow \quad \hat{a}(\epsilon) \equiv \hat{a}_{aft} = A_0 \cos\epsilon + B_0 \sin\epsilon$$
$$\frac{\partial^2 \hat{b}}{\partial t^2} = -i\frac{\partial \hat{a}}{\partial t} = -\hat{b} \quad \rightarrow \quad \hat{b}(\epsilon) \equiv \hat{b}_{aft} = C_0 \cos\epsilon + D_0 \sin\epsilon$$

Finally, the 4 coefficients can be determined by imposing the initial conditions and the matching dictated by the coupling constraint:

$$\hat{a}(0) = \hat{a}_{bef}, \quad \hat{b}(0) = \hat{b}_{bef} \quad \rightarrow \quad A_0 = \hat{a}_{bef}, \quad C_0 = \hat{b}_{bef}$$
$$\frac{\partial \hat{a}}{\partial t} = -i\hat{b}, \quad \frac{\partial \hat{b}}{\partial t} = -i\hat{a} \quad \rightarrow \quad B_0 = -iC_0, \quad D_0 = -iA_0$$

Thus, we can write the field operators $\hat{a}_{aft}$ and $\hat{b}_{aft}$ after the action of $\hat{U}$ as

$$\hat{a}_{aft} = \hat{U}^\dagger \hat{a}_{bef} \hat{U} = \hat{a}_{bef} \cos\epsilon - i\hat{b}_{bef} \sin\epsilon \tag{3}$$
$$\hat{b}_{aft} = \hat{U}^\dagger \hat{b}_{bef} \hat{U} = \hat{b}_{bef} \cos\epsilon - i\hat{a}_{bef} \sin\epsilon \tag{4}$$

We can find an interesting relation between the mixing angle ϵ and the detection probability p considered in the first part of the exercise. In fact, if we consider the average number of photons in the $\hat{b}$ mode after the interaction, we get (supposing no photons in the $\hat{b}$ mode before the interaction, i.e. $\langle \hat{b}^\dagger_{bef} \hat{b}_{bef} \rangle = \langle \hat{a}^\dagger_{bef} \hat{b}_{bef} \rangle = \langle \hat{b}^\dagger_{bef} \hat{a}_{bef} \rangle = 0$)

$$\langle n_b \rangle_{aft} = \langle \hat{b}^\dagger_{aft} \hat{b}_{aft} \rangle = \langle \hat{a}^\dagger_{bef} \hat{a}_{bef} \rangle \sin^2\epsilon \simeq N\epsilon^2 \quad \text{if } \epsilon \text{ is very short}$$

From a direct comparison with the expression for the overall probability of clicking considered in the first part, we get immediately that $p = \epsilon^2$ when ϵ is small enough.

2.1 Evaluation of ρ_{click} adopting the mixing Hamiltonian

In order to to write the density matrix ρ_{aft} after the detection process in terms of the Heisenberg picture operators, we start rewriting the initial quantum state ρ_0 of the three states considered above in terms of creation operators $\hat{a}, \hat{a^\dagger}$ acting on the vacuum state.

For the Fock state $|N\rangle \langle N|$, we get immediately

$$\rho_0 = |N\rangle \langle N| = \frac{1}{N!}(\hat{a}^\dagger)^N |vac\rangle \langle vac| (\hat{a})^N$$

For the coherent state, we make use of its decomposition in terms of Fock states:

$$\rho_0 = |coh : \alpha\rangle \langle coh : \alpha| = \left(e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n (\hat{a}^\dagger)^n}{n!} |vac\rangle \right) \left(\sum_{n'=0} \langle vac| \frac{(\alpha^*)^{n'} (\hat{a})^{n'}}{n'!} e^{-|\alpha|^2/2} \right)$$

Finally, for the thermal state we make use of the same decomposition in terms of Fock states:

$$\rho_0 = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} |n\rangle \langle n| = \frac{1}{Z} \sum_{n=0}^{\infty} \frac{e^{-\beta\hbar\omega n}}{n!} (\hat{a}^\dagger)^n |vac\rangle \langle vac| (\hat{a})^n$$

At this point, ρ_{aft} can be obtained for each case by simply applying the evolution operator on both sides of the density matrix, $\rho_{aft} = \hat{U}\rho_0\hat{U}^\dagger$. Given the explicit expressions of ρ_0 in terms of the $\hat{a}$ and $\hat{b}$ operators and given that $\hat{U}$ acts trivially on the vacuum $\hat{U} |vac\rangle = |vac\rangle$, it is easy to see that ρ_{aft} is obtained by substituting the explicit form of the evolved field operators $\hat{a}_{aft}$ and $\hat{b}_{aft}$ inside the 3 types of density matrix: for example, for the Fock state we get

$$\begin{aligned}
\rho_{click}[fock] &= \hat{U}\rho_0\hat{U}^\dagger = \hat{U}\frac{1}{N!}(\hat{a}^\dagger)^N |vac\rangle \langle vac| (\hat{a})^N \hat{U}^\dagger = \\
&= \frac{1}{N!}(\hat{U}\hat{a}^\dagger\hat{U}^\dagger)^N \hat{U} |vac\rangle \langle vac| \hat{U}^\dagger (\hat{U}\hat{a}\hat{U}^\dagger)^N = \\
&= \frac{1}{N!}(\hat{a}^\dagger_{bef} \cos\epsilon - i\hat{b}^\dagger_{bef} \sin\epsilon)^N |vac\rangle \langle vac| (\hat{a}_{bef} \cos\epsilon + i\hat{b}_{bef} \sin\epsilon)^N
\end{aligned}$$

The substitution in the other 2 cases is exactly the same.

Now, in the small p limit, we can project the density matrix ρ_{aft} on the one-photon state of the $\hat{b}$ mode $|1_b\rangle$ and see what happens. Small p means, first of all, that the probability of multiple clicks, namely of having more than one photon in the b mode is negligible. We can convince ourselves that this is true by looking at the different contributes in the density matrix ρ_{aft}. In particular, the terms $|2_b\rangle \langle 2_b|$, $|3_b\rangle \langle 3_b|$ exhibit a higher power coefficent ϵ in front of them (to the fourth, to the sixth...): thus, we can neglect all such terms if we stop our statistical evaluation at the first/second order of ϵ! Going back to our Fock state, keeping just the 1 photon term in the $\hat{b}$ mode means applying $N-1$ times the ladder operators $\hat{a}$ and then applying one single $\hat{b}$. Taking into account of the degeneracy N of these contributes and simplifying the N factorial, we obtain

$$\langle 1_b| \rho_{click}[fock] |1_b\rangle \simeq \frac{N^2\epsilon^2 |N-1\rangle_a \langle N-1|_a}{N} = N\epsilon^2 |N-1\rangle_a \langle N-1|_a$$

This result matches the one predicted by using (1) in the first part of the exercise, apart from the coefficient $N\epsilon^2$. In reality also this coefficient is perfectly coherent with our new description, because now the photodetector in principle could reveal more than one photon per time, but with a much less probability. This coefficient tells us what is the probability of observing this 1-photon density matrix after having performed the short interaction which weakly couples the cavity with the detector. Moreover, this coefficient was also present in the first part of the exercise, when we said at the beginning that the overall probability of clicking *one* photon was exactly $P = p\langle N\rangle_0$. So, everything is consistent and we are happy.

In a similar way, we can verify that also for the other 2 cases we obtain expressions in accordance with the previous one calculated:

$$\langle 1_b| \rho_{click}[coh] |1_b\rangle \simeq \left(e^{-|\alpha|^2/2} \sum_{n=1}^{\infty} \frac{n\alpha^n (\hat{a}^\dagger)^n}{n\sqrt{(n-1)!}} |n-1\rangle_a \right) \times$$
$$\times \left(\sum_{n'=1} \langle n'-1|_a \frac{n'(\alpha^*)^{n'}(\hat{a})^{n'}}{n'\sqrt{(n'-1)!}} e^{-|\alpha|^2/2} \right) \epsilon^2 = |\alpha|^2\epsilon^2 |coh:\alpha\rangle \langle coh:\alpha|$$

Please notice that for the coherent state the initial average number of photons was equal to the square modulus of its amplitude, i.e. $N = |\alpha|^2$.

$$\langle 1_b| \rho_{click}[therm] |1_b\rangle \simeq \epsilon^2 \frac{1}{Z} \sum_{n=1}^{\infty} \frac{n^2 e^{-\beta\hbar\omega n}}{n} |n-1\rangle_a \langle n-1|_a =$$
$$= N\epsilon^2 \frac{e^{-\beta\hbar\omega}}{N^2} \sum_{n=1}^{\infty} n e^{-\beta\hbar\omega n} |n-1\rangle_a \langle n-1|_a$$

We conclude that also these 2 results show the same ρ_{click} evaluated in the first part of the exercise, multiplied by the overall probability of detecting just one photon during the measure, which is $P \simeq N\epsilon^2$ in the limit of very short ϵ.