

### Adiabatic motion in a time-dependent harmonic potential

Consider the Hamiltonian of the single-particle Harmonic oscillator in one dimension, with a time-dependent equilibrium position  $x_0(t)$ :

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 [\hat{x} - x_0(t)]^2 \quad (1)$$

1. Write the Hamiltonian in the representation of the annihilation,  $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} + \frac{i}{m\omega} \hat{p} \right)$ , and creation operators for the harmonic oscillator centered in  $x = 0$ . What is the coupling induced by the position shift in the Fock basis of the harmonic oscillator centered in  $x = 0$ ?
2. Prove that a coherent state  $|\text{coh}:\alpha_0\rangle = e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle$  satisfies  $\hat{a} |\text{coh}:\alpha_0\rangle = \alpha_0(t) |\text{coh}:\alpha_0\rangle$ , and that the instantaneous ground state of the system at time  $t$  is a coherent state, and determine the value of  $\alpha_0(t)$ . (Hint: shift the annihilation and creation operators by a constant)
3. Prove that the time evolution generated by the Hamiltonian (1) evolves coherent states into coherent states, and derive an equation for  $\alpha(t)$ . (Hint: write the equation of motion for  $\hat{a}$  in the Heisenberg picture.)
4. Prove that, if the derivative of  $x_0(t)$  is negligible and the system starts in the ground state, then it adiabatically follows the instantaneous ground state  $\alpha_0(t)$ . Write a quantitative condition for  $x_0(t)$  to be adiabatic.
5. Assume that

$$x_0(t) = \begin{cases} 0 & t < 0 \\ Xf(t/T) & t \in [0, T] \\ X & t > T \end{cases} \quad (2)$$

with  $f(0) = 0$  and  $f(1) = 1$ , and find, at the first nonzero order in  $1/(\omega T)$ , an estimate for the deviation  $|\alpha_0(T) - \alpha(T)|^2$  at the end of the evolution. How does the power-law scaling depend on the smoothness of  $x_0(t)$ ? Explicitly compute the deviation for  $f(s) = s$ ,  $f(s) = s^2(3 - 2s)$ .

6. Comment on the deviation at intermediate times, how does it scale with  $\omega T$ ?