Solution to: Adiabatic motion in a time-dependent harmonic potential

1. Exploiting the commutation relations for the annihilation and creation operators, one gets

$$\hat{p}^2 = -\frac{\hbar m\omega}{2}\left(\hat{a}^\dagger\hat{a}^\dagger - 2\hat{a}^\dagger\hat{a} + \hat{a}\hat{a} - 1\right) \qquad \hat{x}^2 = \frac{\hbar}{2m\omega}\left(\hat{a}^\dagger\hat{a}^\dagger + 2\hat{a}^\dagger\hat{a} + \hat{a}\hat{a} + 1\right) \tag{1}$$

   and defining the harmonic oscillator length $a_{\text{ho}} = \sqrt{\frac{\hbar}{m\omega}}$ one gets

$$\hat{H}(t) = \hbar\omega\left[\hat{a}^\dagger\hat{a} + \frac{1}{2} - \frac{x_0(t)}{\sqrt{2}a_{\text{ho}}}\left(\hat{a}^\dagger + \hat{a}\right)\right] + \frac{1}{2}m\omega^2 x_0^2(t). \tag{2}$$

   The effect of the shift $x_0(t)$ is to couple sectors which differ by only one excitation, while the last term is proportional to the identity operator, therefore it only imprints a global phase to the state.

2. Applying the annihilation operator to $|\text{coh:}\alpha_0\rangle$ one gets

$$\begin{aligned}\hat{a}\,|\text{coh:}\alpha_0\rangle &= e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}\hat{a}\,|n\rangle = e^{-|\alpha_0|^2/2}\sum_{n=1}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}\sqrt{n}\,|n-1\rangle = \\ &= e^{-|\alpha_0|^2/2}\alpha_0\sum_{n'=0}^{\infty}\frac{\alpha_0^{n'}}{\sqrt{n'!}}\,|n'\rangle = \alpha_0\,|\text{coh:}\alpha_0\rangle\,.\end{aligned} \tag{3}$$

   The ground state of the Hamiltonian $\hat{H}(t)$ is the usual ground state of the harmonic oscillator if the position is shifted as $\hat{\hat{x}} = \hat{x} - x_0(t)$, which implies that one can define shifted annihilation and creation operators from $\hat{\hat{a}} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{\hat{x}} + \frac{i}{m\omega}\hat{p}\right)$

$$\hat{\hat{a}} = \hat{a} - \frac{x_0(t)}{\sqrt{2}a_{\text{ho}}} \qquad \hat{\hat{a}}^\dagger = \hat{a}^\dagger - \frac{x_0(t)}{\sqrt{2}a_{\text{ho}}} \tag{4}$$

   which still satisfy the correct commutation relations $[\hat{\hat{a}}, \hat{\hat{a}}^\dagger] = \mathbb{1}$. The Hamiltonian then reads

$$\hat{H}(t) = \hbar\omega\left(\hat{\hat{a}}^\dagger\hat{\hat{a}} + \frac{1}{2}\right) \tag{5}$$

   and the ground state of the system is the one containing no excitations

$$\hat{\hat{a}}\,|\psi_0\rangle = 0 \implies \hat{a}\,|\psi_0\rangle = \frac{x_0(t)}{\sqrt{2}a_{\text{ho}}}\,|\psi_0\rangle\,. \tag{6}$$

   The ground state is therefore a coherent state with

$$\alpha_0(t) = \frac{x_0(t)}{\sqrt{2}a_{\text{ho}}} \tag{7}$$

   which is expected since a coherent state is a shifted gaussian.

3. In the Heisenberg picture, one can write the equation of motion for the time-dependent annihilation operator $\hat{a}_H(t) = \hat{U}^\dagger(t)\hat{a}\hat{U}(t)$ as

$$\partial_t \hat{a}_H(t) = \frac{i}{\hbar}[\hat{H}_H(t), \hat{a}_H(t)] = -i\omega \hat{a}_H(t) + \frac{i\omega}{\sqrt{2}a_{\text{ho}}} x_0(t) \tag{8}$$

which means that

$$\hat{a}_H(t) = e^{-i\omega t}\left[\hat{a} + \frac{i\omega}{\sqrt{2}a_{\text{ho}}}\int_0^t dt' x_0(t') e^{i\omega t'}\right] \tag{9}$$

therefore, acting on the (time-independent) state $|\text{coh:}\alpha_0(0)\rangle$ in the Heisenberg picture

$$\hat{a}_H(t)\,|\text{coh:}\alpha_0(0)\rangle = \hat{U}^\dagger(t)\hat{a}\hat{U}(t)\,|\text{coh:}\alpha_0(0)\rangle = e^{-i\omega t}\left[\alpha(0) + \frac{i\omega}{\sqrt{2}a_{\text{ho}}}\int_0^t dt' x_0(t') e^{i\omega t'}\right]|\text{coh:}\alpha_0(0)\rangle\,. \tag{10}$$

If $|\psi(t)\rangle = \hat{U}(t)\,|\alpha(0)\rangle$ is the evolved state in the Schrödinger picture, then

$$\begin{aligned}
\hat{a}\,|\psi(t)\rangle =& \hat{U}(t)\hat{a}_H(t)\hat{U}^\dagger(t)\,|\psi(t)\rangle = \hat{U}(t)\hat{a}_H(t)\,|\text{coh:}\alpha_0(0)\rangle = \\
=& e^{-i\omega t}\left[\alpha(0) + \frac{i\omega}{\sqrt{2}a_{\text{ho}}}\int_0^t dt' x_0(t') e^{i\omega t'}\right]\hat{U}(t)\,|\text{coh:}\alpha_0(0)\rangle = \\
=& e^{-i\omega t}\left[\alpha(0) + \frac{i\omega}{\sqrt{2}a_{\text{ho}}}\int_0^t dt' x_0(t') e^{i\omega t'}\right]|\psi(t)\rangle
\end{aligned} \tag{11}$$

which is a coherent state with

$$\alpha(t) = e^{-i\omega t}\left[\alpha(0) + \frac{i\omega}{\sqrt{2}a_{\text{ho}}}\int_0^t dt' x_0(t') e^{i\omega t'}\right]. \tag{12}$$

4. Integrating by part equation (12), and imposing $\alpha(0) = \frac{x_0(0)}{\sqrt{2}a_{\text{ho}}}$, one gets that

$$\alpha(t) = \alpha_0(t) - e^{-i\omega t}\frac{1}{\sqrt{2}a_{\text{ho}}}\int_0^t dt' \dot{x}_0(t') e^{i\omega t'}. \tag{13}$$

Assuming that the speed at which the minimum shifts is always smaller than v in $[0, t]$, then

$$e^{-i\omega t}\frac{1}{\sqrt{2}a_{\text{ho}}}\int_0^t dt' \dot{x}_0(t') e^{i\omega t'} \leq e^{-i\omega t}\frac{v}{\sqrt{2}a_{\text{ho}}}\int_0^t dt' e^{i\omega t'} = -i\frac{v}{\sqrt{2}\omega a_{\text{ho}}}\left(1 - e^{-i\omega t}\right). \tag{14}$$

Therefore, the dynamics is adiabatic if

$$\frac{\dot{x}_0(t)}{a_{\text{ho}}} \ll \omega \quad \forall t \tag{15}$$

5. Inserting the ramp profile into equation (12), and using the dimensionless variable $s = t'/T$ one gets

$$|\alpha_0(t) - \alpha(t)|^2 = \frac{X^2}{2a_{\text{ho}}^2} \left| \int_0^{t/T} ds f'(s) e^{i\omega T s} \right|^2 \tag{16}$$

where $f'(s) = \partial_s f(s)$.

By repeated integration by parts, this can be written as

$$|\alpha_0(t) - \alpha(t)|^2 = \frac{X^2}{2a_{\text{ho}}^2} \left| \sum_{n=1}^{M} (-1)^n \frac{\left[f^{(n)}(s) e^{i\omega T s}\right]_0^{t/T}}{(i\omega T)^n} + \frac{(-1)^{M+1}}{(i\omega T)^{M+1}} \int_0^{t/T} ds f^{(M+1)}(s) e^{i\omega T s} \right|^2 . \tag{17}$$

This means that if the first M derivatives of $x_0(t)$ are continuous, i.e. equal to 0 for $t = 0, T$, then the displacement at the final time T scales as

$$|\alpha_0(T) - \alpha(T)|^2 \sim (\omega T)^{-2(M+1)}. \tag{18}$$

If $f(s) = s$, then (16) at $t = T$ evaluates to

$$|\alpha_0(T) - \alpha(T)|^2 = \frac{4X^2}{(\omega T)^2} \sin^2\left(\frac{\omega T}{2}\right). \tag{19}$$

For $f(s) = s^2(3 - 2s)$ one gets

$$|\alpha_0(T) - \alpha(T)|^2 = \frac{18X^2}{a_{\text{ho}}^2} \frac{1}{(\omega T)^4} \left[2\left[1 + \cos(\omega T)\right] + \frac{8}{\omega T} \sin(\omega T) + \right. \\ \left. + \frac{8}{(\omega T)^2} (1 - \cos(\omega T)) \right]. \tag{20}$$

Notice how, on top of a power-law scaling, there are some interference effects at the frequency of the transition to the excited state ω, which modulate the adiabaticity of the displacement.

The squared overlap between two coherent states is

$$|\langle \alpha | \beta \rangle|^2 = e^{-|\alpha - \beta|^2} \tag{21}$$

meaning that the quantity $|\alpha_0(T) - \alpha(T)|^2$ is minus the logarithm of the usual fidelity.

6. At intermediate times, for $f(s) = s$, one gets

$$|\alpha_0(t) - \alpha(t)|^2 = \frac{4X^2}{(\omega t)^2} \sin^2\left(\frac{\omega t}{2}\right). \tag{22}$$

For $f(s) = s^2(3 - 2s)$:

$$\begin{aligned}|\alpha_0(t) - \alpha(t)|^2 = \frac{18X^2}{a_{\text{ho}}^2}\frac{1}{(\omega T)^2}\Bigg[&\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3 + \left(\frac{t}{T}\right)^4 + 2\frac{\left(\frac{t}{T}\right)^2 - \left(\frac{t}{T}\right)}{\omega T}\sin(\omega T) + \\ &+ \frac{2}{(\omega T)^2}\left[1 + \left(2\left(\frac{t}{T}\right)^2 - 1\right)\cos(\omega t)\right] + \frac{8}{(\omega T)^3}\left(\frac{t}{T}\right)\sin(\omega t) + \\ &+ \frac{8}{(\omega T)^4}(1 - \cos(\omega t))\Bigg].\end{aligned} \tag{23}$$

At intermediate times during the evolution, in both cases there is a $1/(\omega T)^2$ scaling, and only at the final time, where the derivative actually goes to 0 for the second ramp, there is a cancellation and the scaling becomes $1/(\omega T)^4$ as expected.