Basics of path integrals - added 18-19

generic one-particle $\quad H = \frac{P^2}{2m} + V(R)$

$$|\psi(t_f)\rangle = e^{-iH(t_f-t_i)/\hbar}\,|\psi(t_i)\rangle$$

$$= \left(e^{-iH\Delta t/\hbar}\right)^N |\psi(t_i)\rangle$$

with $t_f - t_i = N\,\Delta t$

$$= e^{-iH\Delta t/\hbar}\int dr_1\,|r_1\rangle\langle r_1|\,e^{-iH\Delta t/\hbar}\cdot\int dr_2\,|r_2\rangle\langle r_2|$$

$$\cdot\;\cdots\cdots\;\int dr_{N-1}\,|r_{N-1}\rangle\langle r_{N-1}|e^{-iH\Delta t/\hbar}|\psi(t_i)\rangle$$

$$\langle r|e^{-iH\Delta t/\hbar}|r'\rangle \simeq \langle r|\,e^{-iV(r)\Delta t/\hbar}\,e^{-iP^2\Delta t/2m\hbar}\,|r'\rangle$$

$$= e^{-iV(r)\Delta t/\hbar}\cdot\underbrace{\langle r|\,e^{-iP^2\Delta t/2m\hbar}\,|r'\rangle}_{\text{propagator } G_0(r,r';\Delta t)} = *$$

$$G_0(r,r';\Delta t) = \int\frac{dp}{2\pi}\frac{dp'}{2\pi}\,\langle r|p\rangle\langle p|e^{-iP^2\Delta t/2m\hbar}|p'\rangle\langle p'|r'\rangle$$

$$= \int\frac{dp}{2\pi}\,e^{ipr/\hbar}\,e^{-ipr'/\hbar}\cdot e^{-ip^2\Delta t/2m\hbar}$$

$$= c^{te}\cdot e^{im(r-r')^2/2\Delta t\hbar}$$

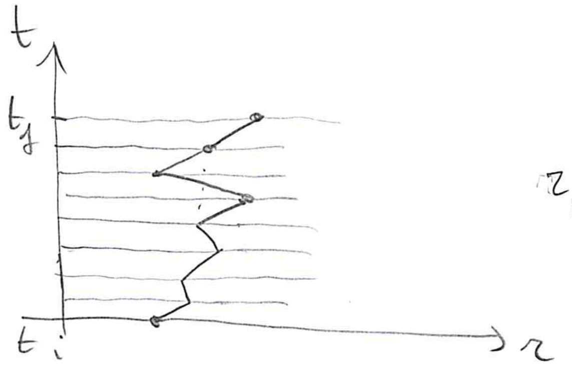
$$* = c^{te}\,e^{i\left(m(r-r')^2/2\Delta t - V(r)\Delta t\right)/\hbar} = G(r,r';\Delta t)$$

$$\langle r_f | \psi(t_f)\rangle = \int dr_1 \dots dr_{N-1}\; G(r_f, r_1; \Delta t)\cdot G(r_1, r_2; \Delta t)\dots$$

$$\dots G(r_{N-1}, r_i, \Delta t)\cdot \langle r_i | \psi(t_i)\rangle$$

$$= c^{te}\cdot\int dr_1 \dots dr_{N-1}\; e^{i\left(m(r_f-r_1)^2/2\Delta t - V(r_f)\Delta t\right)/\hbar}\cdot$$

$$\dots e^{i\left(m(r_1-r_i)^2/2\Delta t - V(r_1)\Delta t\right)/\hbar}\;\psi(r_i;t_i)$$



$$r_{j+1} - r_j \simeq \dot{r}_{j+1}\,\Delta t$$

$$\psi(r_f,t_f) \simeq c^{te}\int dr_1 \dots dr_{N-1}\; e^{i\left(m\dot{r}_f^2/2 - V(r_f)\right)\Delta t/\hbar}\cdot$$

$$\dots e^{i\left(m\dot{r}_1^2/2 - V(r_1)\right)\Delta t/\hbar}\;\psi(r_i;t_i)$$

$$\simeq c^{te}\int dr_1 \dots dr_{N-1}\; e^{i\mathcal{L}(r_f,\dot{r}_f)\Delta t/\hbar}\dots e^{i\mathcal{L}(r_1,\dot{r}_1)\Delta t/\hbar}\cdot$$

$$\cdot\,\psi(r_i,t_i)$$

where $\mathcal{L}(r,\dot{r})$ = Lagrangian corresponding to H

Going from segments to continuous path

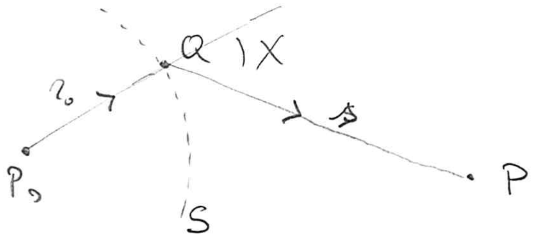
$$\simeq c^{te}\int_{\substack{r(t_i)=r_i\\ r(t_f)=r_f}} \mathcal{D}r(t)\;\underbrace{e^{i\int_{t_i}^{t_f}\mathcal{L}(r(t),\dot{r}(t))\,dt/\hbar}}_{e^{iS/\hbar}}$$

Amplitude to be at r_f at time t_f is sum over all paths from r_i at t_i weighted with (complex) amplitude given by classical action S.

Classical limit $\hbar \to 0$ $\Rightarrow$ only saddle-point contributes where action is $\underbrace{\text{extremum}}$

classical Euler-Lagrange equations.

Analogy with Huygens and Kirchhoff principles in wave propagation:



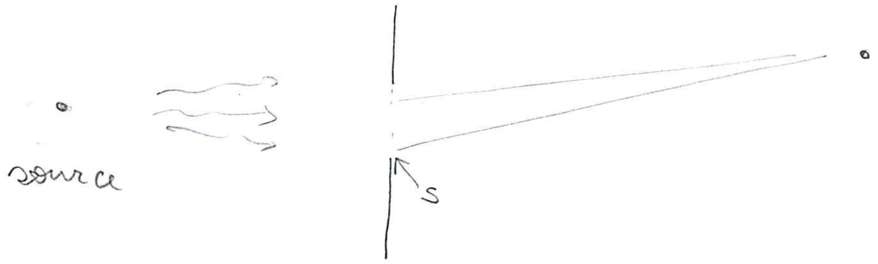
H) $$U(P) = -\frac{i}{\lambda}\, U(r_0) \int_S \frac{e^{iks}}{s}\, K(\chi)\, dS$$

ex. Arago-Poisson bright spot.

K) $$U(P) = \frac{1}{4\pi} \int_S \left[U \frac{\partial}{\partial n}\left(\frac{e^{iks}}{s}\right) - \frac{e^{iks}}{s} \frac{\partial U}{\partial n} \right] dS$$

for monochromatic light and standard wave eq.

$$\simeq -\frac{i}{2\lambda} \int_S dS \; a(r) \, \frac{e^{iks}}{s} \left(\cos(\hat{n s}) - \cos(\hat{n r}) \right)$$



field in aperture unperturbed
$\sim a(r)\, e^{ikr}$

Advantages over canonical quantization

- $\mathcal{L}$ is relativistic scalar $\longrightarrow$ easier covariance
- avoid "mysterious" concept of momentum
- gauge fields easily included
- role of r and t more symmetric [Lenić-ic, PRA 2015]

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References:

Feynman - Hibbs "Quantum Mechanics and Path Integrals", 1965.
Albeverio - Hoegh-Krohn - Mazzucchi "Mathematical theory of Feynman Path Integrals", Springer 2008