Lecture 1 : Maxwell eqs and wave propagation

$$\begin{cases} \underline{\nabla} \cdot \underline{E} = 4\pi \rho \\ \underline{\nabla} \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \\ \nabla \cdot \underline{B} = 0 \\ \nabla \times \underline{B} = \frac{1}{c}\frac{\partial \underline{E}}{\partial t} + \frac{4\pi}{c}\underline{J} \end{cases}$$

* in Fourier space :

$$f(\underline{r},t) \longrightarrow \hat{f}(\underline{k},\omega) = \int d^3r \, dt \, e^{-i(\underline{k}\cdot\underline{r}-\omega t)} f(\underline{r},t)$$

inverse : $f(\underline{r},t) = \int \frac{d^3\underline{k}}{(2\pi)^3} \frac{d\omega}{2\pi} e^{i(\underline{k}\cdot\underline{r}-\omega t)} \hat{f}(\underline{k},\omega)$

the same holds for vectorial functions :

$$\underline{f}(\underline{r},t) = \{ f_x(\underline{r},t), f_y(\underline{r},t), f_z(\underline{r},t) \} .$$

* differential operators

$$\begin{array}{ccc} \underline{f}(\underline{r},t) & \longrightarrow & \underline{\nabla}\cdot\underline{f} \\ \Big\downarrow & & \\ \hat{\underline{f}}(\underline{k},\omega) & \longrightarrow & i\underline{k}\cdot\hat{\underline{f}} \end{array}$$

Analogously : $\partial_t \longrightarrow -i\omega$

$\underline{\nabla}\times \longrightarrow i\underline{k}\times$

...

$$\begin{cases} i\underline{k}\cdot\underline{\tilde{E}} = 4\pi\tilde{\rho} \\ i\underline{k}\times\underline{\tilde{E}} = i\dfrac{\omega}{c}\underline{\tilde{B}} \\ i\underline{k}\cdot\underline{\tilde{B}} = 0 \\ i\underline{k}\times\underline{\tilde{B}} = -i\dfrac{\omega}{c}\underline{\tilde{E}} + \dfrac{4\pi}{c}\underline{\tilde{J}} \end{cases}$$

__exercise__: prove charge conservation from Maxwell in Fourier:

$$\underline{k}\cdot(\underline{k}\times\underline{\tilde{B}}) = 0$$

$$\|$$

$$\underline{k}\cdot\left(-\frac{\omega}{c}\underline{\tilde{E}} + \frac{4\pi}{c}i\,\tilde{J}\right)$$

$$\|$$

$$\frac{i\omega}{c}\,i\underline{k}\cdot\underline{\tilde{E}} - \frac{4\pi}{c}\,i\underline{k}\cdot\tilde{J}$$

$$\|$$

$$\frac{i\omega}{\cancel{c}}\,4\pi\tilde{\rho} - \frac{4\pi}{\cancel{c}}\,i\underline{k}\cdot\tilde{J}$$

$$\Longrightarrow\quad i\underline{k}\cdot\tilde{J} - i\omega\tilde{\rho} = 0$$

$$\boxed{\nabla\cdot J + \frac{\partial\rho}{\partial t} = 0}$$

* Split longitudinal / transverse components:

$$\tilde{\underline{E}}(\underline{k},\omega) = \underbrace{\frac{\underline{k}\,(\underline{k}\cdot\tilde{\underline{E}})}{k^2}}_{\substack{\text{longitudinal} \\ /\!/\, \underline{k}}} + \underbrace{\left(\tilde{\underline{E}} - \frac{\underline{k}\,(\underline{k}\cdot\tilde{\underline{E}})}{k^2}\right)}_{\text{transverse } \perp\, \underline{k}}$$

$$\tilde{E}_L = P_L \cdot \tilde{\underline{E}} \qquad\qquad P_T\, \tilde{\underline{E}} = \tilde{E}_T$$

$$i k \cdot \tilde{\underline{E}}_L(\underline{k},\omega) = 4\pi\, \tilde{\rho}(\underline{k},\omega)$$

$$\underline{k} \times \tilde{\underline{E}}_T(\underline{k},\omega) = \frac{\omega}{c}\, \tilde{B}_T(\underline{k},\omega)$$

$$k \cdot \tilde{B}_L(\underline{k},\omega) = 0$$

$$\underline{k} \times \tilde{B}_T(\underline{k},\omega) = -\frac{\omega}{c}\tilde{E}_T + \frac{4\pi}{c}\tilde{J}_T + \underbrace{\cancel{-\frac{\omega}{c}\tilde{E}_L + \frac{4\pi}{c}\tilde{J}_L}}_{0 \text{ by charge conservation!}}$$

Longitudinal and transverse sectors <u>decoupled</u>!

$$\left.\begin{aligned} ik\,\tilde{E}_L &= 4\pi\tilde{\rho} \\ ik\,\tilde{B}_L &= 0 \end{aligned}\right\} \qquad \left\{\begin{aligned} \underline{k}\times\tilde{E}_T &= \frac{\omega}{c}\tilde{B}_T \\ \underline{k}\times\tilde{B}_T &= -\frac{\omega}{c}\tilde{E}_T + \frac{4\pi}{c}\tilde{J}_T \end{aligned}\right.$$

<u>NOTE</u>: $\tilde{J}_T$ and ρ unrelated!

Linearity of Maxwell eqs:

given source $\{\rho, \underline{J}\}$:

$$\begin{Bmatrix} E \\ B \end{Bmatrix} = \begin{Bmatrix} E \\ B \end{Bmatrix}_{source} + \text{any} \begin{Bmatrix} E \\ B \end{Bmatrix}_{free.}$$

free solution:

$$\left.\begin{aligned} ik\,\tilde{E}_L &= 0 \\ ik\,\tilde{B}_L &= 0 \end{aligned}\right\} \Longrightarrow \tilde{E}_L = \tilde{B}_L = 0$$

$$\begin{cases} \underline{k} \times E_T = \dfrac{\omega}{c} B_T \\ \underline{k} \times B_T = -\dfrac{\omega}{c} E_T \end{cases}$$

$$\underline{k} \times B_T = \underline{k} \times \left(\frac{c}{\omega}\, \underline{k} \times E_T\right) = -\frac{\omega}{c} E_T$$

on transverse subspace $\underline{k} \times (\underline{k} \times \quad) = -k^2\, \mathrm{Id}|_T$

[indeed $\underline{k} \times (\underline{k} \times \underline{v}) = k\,(\underline{k} \cdot \underline{v}) - (\underline{k} \cdot \underline{k}) \cdot v$

$$-k^2\, \underline{E}_T = -\frac{\omega^2}{c^2} E_T$$

$$\boxed{\left(\frac{\omega^2}{c^2} - \underline{k}^2\right) \tilde{\underline{E}}_T = 0}$$

Wave equation for transverse e.m. waves

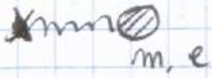
$$\underline{\tilde{E}}_T(\underline{k},\omega) = \sum_k \underline{\tilde{E}}_T^{(i)} \, e^{i(\underline{k}_i \cdot \underline{x} - \omega t)}$$

with $\underline{\tilde{E}}_T^{(i)} \perp \underline{k}_i$

* for each $\underline{k} \longrightarrow 2$ polarization states $\perp \underline{k}$.
* frequency $\omega = c \cdot |\underline{k}|$

Drude - Lorentz model of dielectric

Harmonic oscillator



$$m\ddot{\underline{r}} = e \cdot E(\underline{r},t) - m\gamma\dot{\underline{r}} - m\omega_0^2 r$$

Newton's law — electric force — friction — harmonic restoring force

* Harmonic electric field at ω
* Assume $E(r,t) \simeq E(r=0,t)$

$$-m\omega^2 \underline{r}(\omega) = e\,\underline{E}(\omega) + i\omega m\gamma\underline{r}(\omega) - m\omega_0^2 \underline{r}$$

$$r(\omega) = \frac{e\,\underline{E}(\omega)}{m\omega_0^2 - m\omega^2 - i m\gamma\omega}$$

$$\underline{d} = e\,r = \frac{\frac{e^2}{m}\,\underline{E}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

Polarization density $N\underline{d} = \underbrace{\frac{N\frac{e^2}{m}}{\omega_0^2 - \omega^2 - i\gamma\omega}}_{\text{dielectric susceptibility } \chi(\omega)}\,\underline{E}$

density of atoms

$$\boxed{\varepsilon(\omega) = 1 + \frac{4\pi N e^2}{m}\,\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}}$$

$$P(\underline{k}, \omega) = \chi(\omega) \cdot \underline{E}(\underline{k}, \omega)$$

$\hookrightarrow$ does not depend on $\underline{k}$ unless "spatial dispersion" is present, i.e. interactions between harmonic oscillators

$$\tilde{J}(\underline{k}, \omega) = -i\omega\, \chi(\omega)\, E(\underline{k}, \omega) = -i\omega\, \underline{P}(\underline{k}, \omega)$$

indeed: $\quad j = \dot{x} \cdot e \;, \quad J = n e \dot{x} \; \ldots$

$$\tilde{\varrho}(\underline{k}, \omega) = -i\underline{k} \cdot \underline{P}(\underline{k}, \omega)$$

charge conservation:

$$i\underline{k}\, J + \omega \varrho \overset{?}{=} 0$$

$$i\underline{k}\,(-i\omega\, \underline{P}) + \omega\,(-i\underline{k} \cdot \underline{P}) =$$

$$= \omega\,(\underline{k} \cdot \underline{P}) - \omega\,(\underline{k} \cdot \underline{P}) = 0 \qquad \underline{\text{ok}}.$$

Transverse field: $\quad E_L = P_L = 0 \iff \varrho = 0$

$$\underline{k} \times E_T = \frac{\omega}{c} \underline{B}_T$$

$$i\underline{k} \times \underline{B}_T = -i\frac{\omega}{c} E_T + \frac{4\pi}{c} J_T = -i\frac{\omega}{c} E_T + \frac{4\pi}{c}(-i\omega\chi)\, E_T$$

$$= -i\frac{\omega}{c}\left(1 + 4\pi\chi\right) \underline{E}_T$$

$$\left[\frac{\omega^2}{c^2}\left(1+4\pi\chi(\omega)\right) - k^2\right] E_T = 0$$

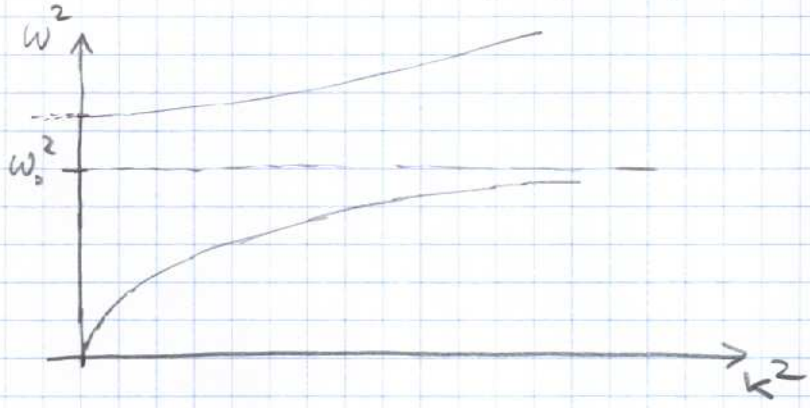
Free wave propagation for

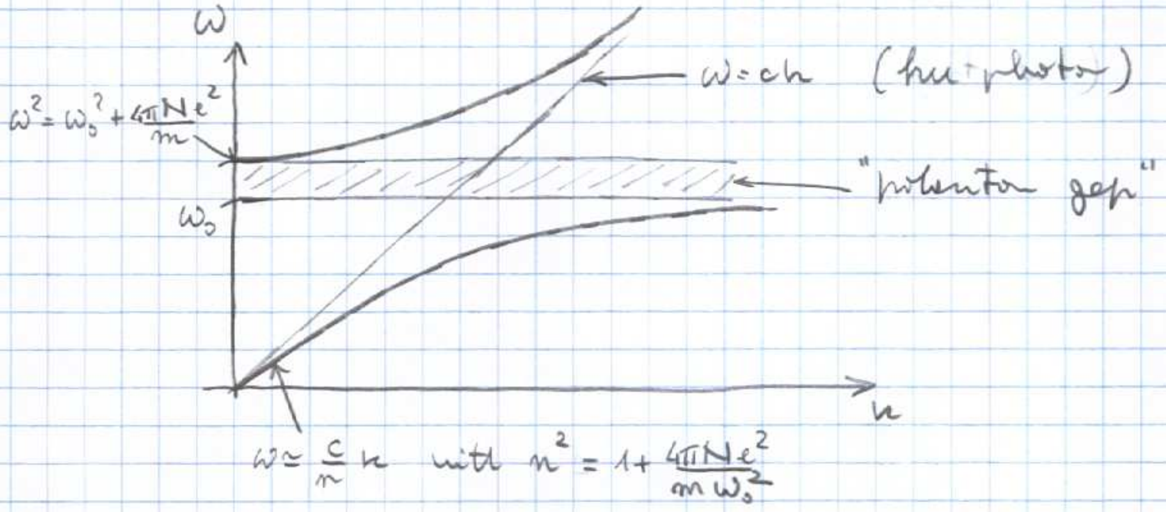
$$\boxed{\frac{\omega^2}{c^2}\,\varepsilon(\omega) - k^2 = 0}$$

$\hookrightarrow$ dispersion relation for e.m. waves in matter.

<u>Drude - Lorentz model</u>, assume $\gamma \to 0$:

$$\frac{\omega^2}{c^2}\,\varepsilon(\omega) = \frac{\omega^2}{c^2}\left(1 + \frac{4\pi N e^2}{m}\,\frac{1}{\omega_0^2 - \omega^2 - i0^+}\right) = k^2$$





* dispersion relation for TRANSVERSE e.m. waves.

↳ POLARITONS

What about LONGITUDINAL waves:

$$\begin{cases} i k E_L = 4\pi \rho \\ i k B_L = 0 \end{cases}$$

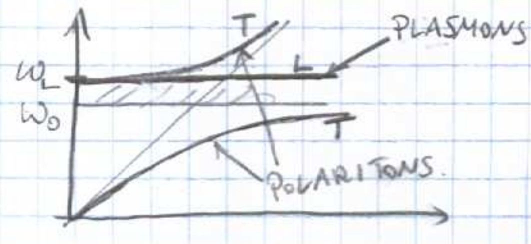
$$\rho = -i \underline{k} \cdot \underline{P} = -i k E_L \cdot \chi(\omega)$$

$$i k E_L = -4\pi i k \chi(\omega) E_L$$

$$i k (1 + 4\pi \chi(\omega)) E_L = 0$$

wave solution for ω such that $1 + 4\pi\chi(\omega) = \varepsilon(\omega) = 0$

↳ PLASMONS



$(\omega_L - \omega_T = \text{LT-splitting})$

What happens in the polariton gap?

* no solution for $\omega, \underline{k} \in \mathbb{R}$

* generalise to complex $\underline{k}$, real ω.

$\hookrightarrow$ interface properties of monochromatic incident wave.

Restricting to transverse waves:

$$\left(\varepsilon(\omega) \cdot \frac{\omega^2}{c^2} + \partial_x^2 \right) E(x, \omega) = 0$$

polariton gap $\varepsilon(\omega) < 0$.

$$E(x, \omega) = E_0 \, e^{i\tilde{k}x}$$

$$\varepsilon(\omega) \, \omega^2/c^2 = \tilde{k}^2 \implies \tilde{k} = \pm i \sqrt{|\varepsilon(\omega)|} \cdot \omega/c$$

* exponentially decaying wave
* but no energy transport (look at $\underline{S} \longrightarrow$ exercise!)

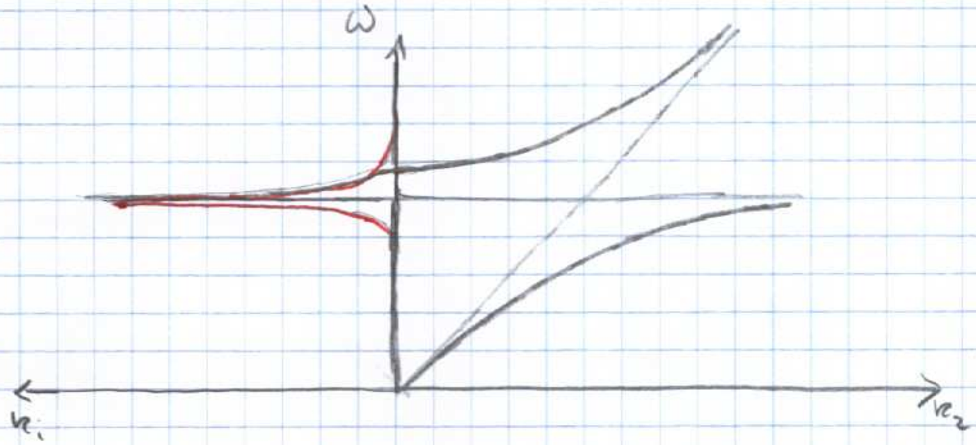
* incident wave totally reflected.

* directly extends to complex $\varepsilon(\omega)$ case

* another family of solutions for real $\underline{k}$, complex ω [Tait, PRB $\underline{5}$, 649 (1972)]

Absorption

$$\frac{\omega^2}{c^2}\,\varepsilon(\omega) = \frac{\omega^2}{c^2}\left[1+\frac{4\pi N e^2}{m}\,\frac{1}{\omega_0^2-\omega^2-i\gamma\omega}\right] = k^2$$



* absorption tail scales as $|\omega-\omega_0|^{-2}$, while refraction scales $|\omega-\omega_0|^{-1}$

$$\varepsilon(\omega)-1 \simeq \frac{2\pi N e^2}{m\omega_0}\,\frac{1}{\omega_0-\omega-i\gamma/2} =$$

$$= \frac{2\pi N e^2}{m\omega_0}\left\{\frac{\omega_0-\omega}{(\omega_0-\omega)^2+\gamma^2/4}+\frac{i\gamma/2}{(\omega_0-\omega)^2+\gamma^2/4}\right\}$$

* low frequency $\quad k = \frac{\omega}{c}\left(1+\frac{4\pi N e^2}{m\omega_0^2}\right)^{1/2} \simeq \frac{\omega}{c}\left(1+\frac{2\pi N e^2}{m\omega_0^2}\right)$

i.e. $n_0 = 1+\frac{4\pi N e^2}{m\omega_0^2}$

refractive index

* dilute vs. dense medium. $\frac{4\pi N e^2}{m\omega_0\gamma} \lessgtr 1$

– dilute:

$$\frac{\omega^2}{c^2}\left[1+\frac{4\pi N e^2}{m}\frac{1}{\omega_0^2-\omega^2-i\gamma\omega}\right]$$

$$\simeq \frac{\omega^2}{c^2}\left[1+\frac{2\pi N e^2}{m\omega_0}\frac{1}{\omega_0-\omega-i\gamma/2}\right]$$

$$\simeq \frac{\omega^2}{c^2}\left[1+\underbrace{\frac{2\pi N e^2}{m\omega_0}\frac{\omega_0-\omega+i\gamma/2}{(\omega_0-\omega)^2+\gamma^2/4}}_{\ll 1 \text{ in a dilute medium}}\right] = k^2$$

$$= k_r^2 - k_i^2 + 2i\,k_r k_i$$

as $k_r \simeq \omega/c$, $k_i \ll k_r \simeq \omega/c$:

$$k_r^2 \simeq \frac{\omega^2}{c^2}\left(1+\frac{2\pi N e^2}{m\omega_0}\frac{\omega_0-\omega}{(\omega_0-\omega)^2+\gamma^2/4}\right)$$

$$k_r \simeq \frac{\omega}{c}\left(1+\frac{\pi N e^2}{m\omega_0}\frac{\omega_0-\omega}{(\omega_0-\omega)^2+\gamma^2/4}\right) \quad \text{close to 1}$$

$$k_i \simeq \frac{\omega^2}{c^2}\cdot\frac{1}{2\cancel{i}}\frac{\cancel{i}\gamma/2}{(\omega_0-\omega)^2+\gamma^2/4}\cdot\frac{2\pi N e^2}{m\omega_0}\cdot\frac{c}{\omega} \simeq$$

$$\simeq \frac{1}{c}\frac{2\pi N e^2}{m}\frac{\gamma/4}{(\omega_0-\omega)^2+\gamma^2/4} \quad \text{peaked at } \omega_0 \text{ with linewidth } \gamma$$

- dense medium : γ is perturbation on polariton picture
(exception for small region $|\omega_0 - \omega| \leq \gamma$)

<u>NOTE</u> : there is <u>NO</u> intrinsic lower bound on γ.
High quality media have $\gamma \to 0$

$\hookrightarrow$ γ is NOT radiative linewidth

Wave propagation : general concepts

dispersion $\quad F(\omega, k) = 0$

$\phi(x, t=0)$ is gaussian wavepacket at $x=0$

$$\phi(x, t=0) = \phi_0 \exp\left(-x^2/2\sigma^2\right) \exp(i k_0 x)$$

$$\tilde{\phi}(k, t=0) = \sqrt{2\pi}\sigma \, \phi_0 \exp\left(-(k-k_0)^2 \sigma^2/2\right)$$

long wavepacket (large σ) $\Longrightarrow$ narrow in $\mathcal{F}$-space around k_0

$$\omega(k) = \omega(k_0) + \frac{d\omega}{dk}(k-k_0) + \frac{1}{2}\frac{d^2\omega}{dk^2}(k-k_0)^2 + \ldots$$

$$\tilde{\phi}(k,t) \simeq \sqrt{2\pi\sigma}\;\phi_0\, \exp\left(-(k-k_0)^2\sigma^2/2\right)\cdot$$
$$\cdot \exp\left(-i\omega_0 t\right)\cdot \exp\left(-i\frac{d\omega}{dk}(k-k_0)\cdot t\right)$$
$$\cdot \exp\left(-i\frac{1}{2}\frac{d^2\omega}{dk^2}(k-k_0)^2\, t\right)\cdot \;\ldots$$

$$\phi(x,t) \simeq \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\sqrt{2\pi\sigma}\;\phi_0\; e^{-i\omega_0 t}\; e^{-(k-k_0)^2\sigma^2/2}\; e^{-i\frac{d\omega}{dk}(k-k_0)t}$$
$$\cdot\, e^{-i\frac{d^2\omega}{dk^2}(k-k_0)^2/2\, t}\;\cdot\; e^{ikx}$$

$$= \sqrt{\frac{\sigma}{2\pi}}\;\phi_0\, e^{-i\omega_0 t}\, e^{ik_0 x}\int_{-\infty}^{\infty} dk\; e^{-(k-k_0)^2\sigma^2/2}\; e^{i(k-k_0)\left(x-\frac{d\omega}{dk}t\right)}$$
$$\cdot\, e^{-i\frac{d^2\omega}{dk^2}(k-k_0)^2\, t/2}$$

if $\frac{d^2\omega}{dk^2}$ negligible : $\phi(x,t) = \phi_0\, \underbrace{e^{ik_0x}\, e^{-i\omega_0 t}}\cdot\, \underbrace{e^{-\left(x-\frac{d\omega}{dk}t\right)^2/2\sigma^2}}$

plane wave moving at ω_0/k_0 — envelope moving at $d\omega/dk$

PHASE VELOCITY vs. GROUP VELOCITY

$\frac{d^2\omega}{dk^2}$ introduces DISPERSION effects : $\frac{d^2\omega}{dk^2} = \frac{d}{dk}\left(\frac{d\omega}{dk}\right) = \frac{d}{dk}v_{gr}$

integral can be performed by closing the square :

$$\frac{1}{2}\left[(k-k_0)^2\left(\sigma^2 + i\frac{d^2\omega}{dk^2}t\right) + 2i\left(x-\frac{d\omega}{dk}t\right)(k-k_0) + \ldots\right]$$

$$\phi(x,t) = \frac{1}{\left(1+i\frac{d^2\omega}{dk^2}\,t/\sigma^2\right)^{1/2}}\; e^{i(k_0 x-\omega_0 t)} \cdot \exp\left[-\frac{(x-v_{gr}t)^2}{2\sigma^2\left(1+i\frac{d^2\omega}{dk^2}\,t/\sigma^2\right)}\right]$$

* wave packet centred at $x(t) = v_{gr}\cdot t$

* length $\bar{\sigma}^2 = \sigma^2 + \frac{1}{\sigma^2}\left(\frac{d^2\omega}{dk^2}\right)^2\cdot t^2$

* phase profile

$$\exp\left[+\frac{i}{2\sigma^2\bar{\sigma}^2}\cdot\frac{d^2\omega}{dk^2}\,t\cdot(x-v_{gr}t)^2\right]$$

local $\bar{k}(x) = \frac{t}{\sigma^2\bar{\sigma}^2}(x-v_{gr}t)\,\frac{d^2\omega}{dk^2} + k_0$

$\frac{d^2\omega}{dk^2}t \gg \sigma$

$$\simeq \frac{1}{\frac{d^2\omega}{dk^2}\cdot t}\,(x-v_{gr}\cdot t) + k_0$$

i.e. $\left(\frac{d^2\omega}{dk^2}(\bar{k}-k_0)\right)\cdot t = (x-v_{gr}t)$

displacement from average v_{gr}

↳ ballistic expansion of wave packet

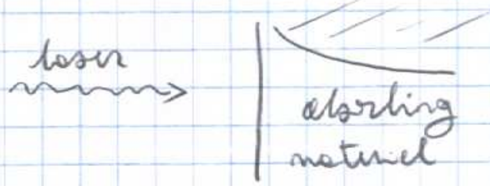
max speed $\frac{d^2\omega}{dk^2}\cdot\frac{1}{\sigma} \Longrightarrow \bar{\sigma}(t) \simeq \frac{1}{\sigma}\,\frac{d^2\omega}{dk^2}$

In presence of absorption $F(\omega, k) = 0$
requires either $\mathrm{Im}(\omega)$ or $\mathrm{Im}(k)$ or both $\neq 0$.

two families of solutions $\rightarrow$ chosen by boundary conditions.

i) $\mathrm{Im}(\omega) = 0, \quad \mathrm{Im}(k) \neq 0.$

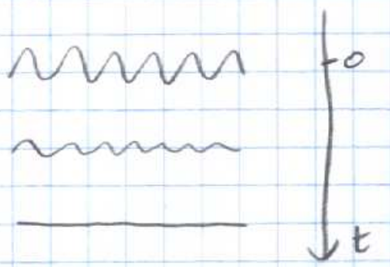
$$\phi(x,t) \simeq \phi_0 \exp(-i\omega t) \exp(i k_r x) \exp(-k_i x)$$



ii) $\mathrm{Im}(\omega) < 0, \quad \mathrm{Im}(k) = 0.$

$$\phi(x,t) = \phi_0 \exp(-i\,\mathrm{Re}(\omega) t) \exp(-|\mathrm{Im}(\omega)|\, t) \exp(i k x)$$

wave in infinite absorbing material: time-evolution



Relation between $\mathrm{Im}(\omega)$ of (ii) and $\mathrm{Im}(k)$ of (i)

$$F(\omega_r - i\omega_i, k_r) = 0 = F(\omega_r, k_r + i k_i)$$

$$F(\omega_r, k_r) - i \frac{\partial F}{\partial \omega} \omega_i \qquad\qquad F(\omega_r, k_r) + i \frac{\partial F}{\partial k} k_i$$

$$\Longrightarrow \omega_i = -\left(\frac{\partial F}{\partial k} \Big/ \frac{\partial F}{\partial \omega}\right) \cdot k_i$$

group velocity:

$$F(\omega_r + \delta\omega_r + i\,\omega_i\,,\; k_r + \delta k_r) = 0$$

$$\Downarrow$$

$$\underbrace{\cancel{F(\omega_r + i\omega_i\,,\, k_r)}}_{0} + \frac{\partial F}{\partial \omega}\,\delta\omega_r + \frac{\partial F}{\partial k}\,\delta k_r$$

$$\Rightarrow \quad \frac{\delta\omega_r}{\delta k_r} = -\frac{\partial F}{\partial k}\Big/\frac{\partial F}{\partial \omega} = v_{gr}$$

$$\boxed{\omega_i = v_{gr}\, k_i}$$

lifetime $\tau = 1/\omega_i$; penetration length $\ell = 1/k_i$

$$\boxed{\ell = v_{gr} \cdot \tau}$$

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