Lecture 2 : Mirrors, waveguides and cavities

Perfect conductor

$B_n = 0$, $\vec{B}_\parallel$ screened by surface current : $\hat{n} \times \vec{B} = \frac{4\pi}{c} \vec{K}$

$\vec{E}_\parallel = 0$, E_n screened by surface charge $E_n = 4\pi\sigma$

σ = surface charge : $\rho(z) = \sigma \cdot \delta(z)$

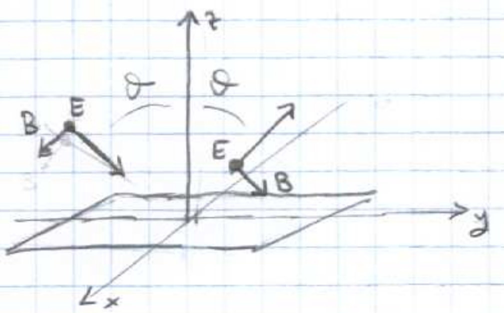
$\vec{K}$ = surface current , $\vec{K} \cdot \hat{n} = 0$, $\vec{J} = \vec{K}\, \delta(z)$

(surface of conductor is at $z=0$)

EXERCISE : show that $\vec{\nabla}_\parallel \cdot \vec{K} + \frac{\partial \sigma}{\partial t} = 0$

and give a physical interpretation.

1- Plane mirror



TE polarization.

$$E_x = E_1 e^{-ik_z z} e^{ik_y y} e^{-i\omega t} + E_2 e^{ik_z z} e^{ik_y y} e^{-i\omega t}$$

$$E_y = 0.$$

2.2

$$B_y = \left(E_1 (-\cos\vartheta)\, e^{-ik_z z} + E_2 \cos\vartheta\, e^{ik_z z}\right) e^{ik_y y} e^{-i\omega t}$$

$$B_z = \left(E_i (-\sin\vartheta)\, e^{-ik_z z} + E_2 (-\sin\vartheta)\, e^{ik_z z}\right) e^{ik_y y} e^{-i\omega t}$$

on the $z=0$ plane:

universal dependence due to translational symmetry along y + time invariance

$$E_x = (E_i + E_2)\, e^{ik_y y} e^{-i\omega t}$$

$$E_y = 0$$

$$B_y = (E_2 - E_i) \cos\vartheta\, e^{ik_y y} e^{-i\omega t}$$

$$B_z = (E_i + E_2)(-\sin\vartheta)\, e^{ik_y y} e^{-i\omega t}$$

Boundary condition: $E_x = 0$ for $z=0 \implies E_i + E_2 = 0$.

implies that also $B_z = 0$.

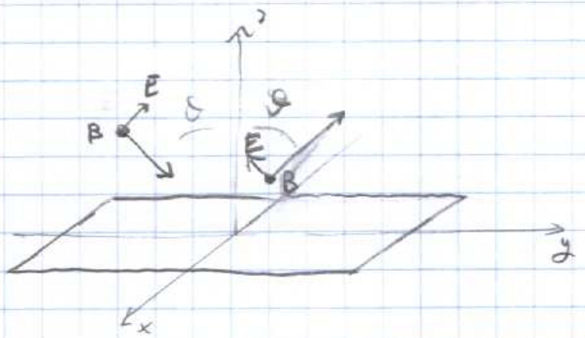
On $z=0$ plane $\longrightarrow$ only $B_y = -2E_i \cos\vartheta\, e^{ik_y y} e^{-i\omega t}$

$$\vec{K} = \frac{c}{4\pi} (-2E_i \cos\vartheta)\, \hat{e}_x \, e^{ik_y y} e^{-i\omega t}$$

$\hookrightarrow$ surface current "created" by incident $\vec{E} /\!/ \hat{e}_x$

current purely transverse $\rightarrow \sigma = 0$ always.

TM polarization



$$E_x = 0 = B_y = B_z$$

$$E_y = \left(E_i \sin\vartheta \; e^{-ik_z z} + E_2 (-\sin\vartheta)\, e^{ik_z z}\right) e^{ik_y y}\, e^{-i\omega t}$$

$$E_z = \left(E_i \cos\vartheta \; e^{-ik_z z} + E_2 \cos\vartheta \; e^{ik_z z}\right) e^{ik_y y}\, e^{-i\omega t}$$

$$B_x = \left(E_1 \, e^{-ik_z z} + E_2 \, e^{ik_z z}\right) e^{ik_y y}\, e^{-i\omega t}$$

Boundary condition on $z=0$ plane:

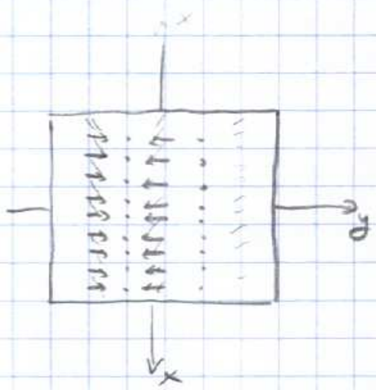
$$E_y(z=0) = 0 \quad \Longrightarrow \quad E_i - E_2 = 0$$

On $z=0$ plane:

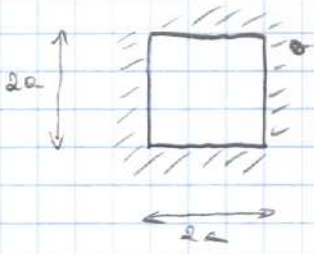
$$E_z = 2E_i \cos\vartheta \; e^{ik_y y}\, e^{-i\omega t}$$

$$B_x = 2E_i \, e^{ik_y y}\, e^{-i\omega t}$$

$$\Longrightarrow \begin{cases} \sigma = \dfrac{2E_i}{4\pi} \cos\vartheta \; e^{ik_y y}\, e^{-i\omega t} \\[2ex] \vec{K} = \hat{e}_y \, \dfrac{2cE_i}{4\pi} \, e^{ik_y y}\, e^{-i\omega t} \end{cases}$$



2 – Waveguide



metallic mirrors.

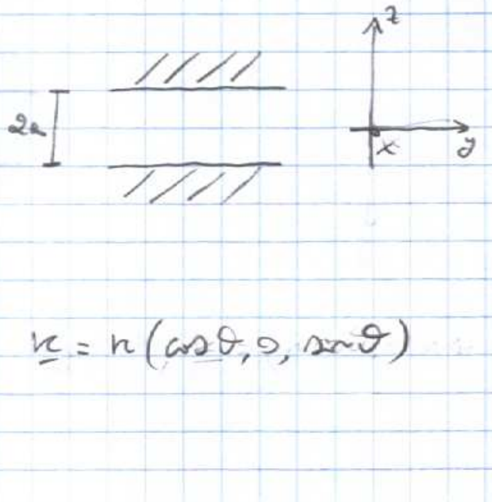
{ cylindrical geometry invariant along z
{ temporal invariance

k_z and ω good "quantum numbers".

Look for combination of plane waves that satisfy boundary conditions on all 4 mirrors.

Simpler, illustrative case:

* 2 planar mirrors



* $k_x, k_y, \omega \rightarrow$ "quantum numbers"

  ↳ for simplicity of notation $\underline{k} = k(\cos\theta, 0, \sin\theta)$

* k_z quantized
* frequency $\omega = c \cdot k$

TE polarization:

$$\underline{E} = \hat{e}_y \cdot \left(E_i \, e^{ik_z z} + E_r \, e^{-ik_z z} \right) e^{ik_x x} \, e^{-i\omega t}$$

$$B_z = \left(E_i \cos\theta \, e^{ik_z z} + E_r \cos\theta \, e^{-ik_z z} \right) e^{ik_x x} \, e^{-i\omega t}$$

$$B_x = \left(E_i \, (-\sin\theta) \, e^{ik_z z} + E_r \sin\theta \, e^{-ik_z z} \right) e^{ik_x x} \, e^{-i\omega t}$$

Boundary conditions at $z = \pm a$

$$E_y(z = \pm a) = 0$$

$$\Rightarrow \begin{cases} E_1 \, e^{ik_z a} + E_2 \, e^{-ik_z a} = 0 \\ E_1 \, e^{-ik_z a} + E_2 \, e^{ik_z a} = 0 \end{cases}$$

non-trivial solution $\Rightarrow \quad e^{2ik_z a} - e^{-2ik_z a} = 0$

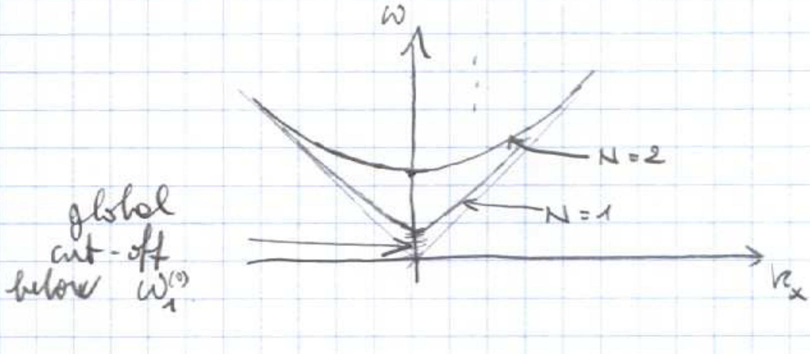
$$\sin(2k_z a) = 0$$

$$2k_z a = N\pi$$

$$\boxed{\text{quantization condition} \quad k_z = \frac{N\pi}{2a}}$$

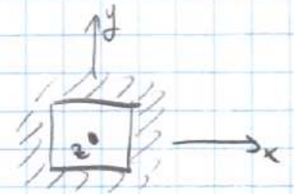
$B_x(z = \pm a) = 0$ automatically satisfied.

$$\boxed{\text{dispersion relation} \quad \omega^2 = c^2(k_z^2 + k_x^2)}$$



for every N $\longrightarrow$ cut-off frequency at $N \cdot \frac{c\pi}{2a} = \omega_N^{(0)}$

then tends to light line for $\omega \gg \omega_N^{(0)}$

Complete case : 

$$\omega_{m,n} = \sqrt{\left(\frac{\pi c m}{2a}\right)^2 + \left(\frac{\pi c n}{2a}\right)^2 + c^2 k_z^2}$$

quantization along both x, y , n or m > 0.

cut-off frequency $\frac{c\pi}{2a} = \omega_{0,1}^{(0)}$

↳ more details in Jackson, "Classical E.D"

3- cavities

* waveguide is closed by metallic mirrors at ends.

* also k_z quantized $k_z = \frac{\pi N_z}{2L_z}$

* discrete frequencies.

$$\omega_{m,n,N_z} = c \cdot \sqrt{\frac{\pi^2}{(2L_z)^2} N_z^2 + \frac{\pi^2}{(2a)^2} m^2 + \frac{\pi^2}{(2a)^2} n^2}$$

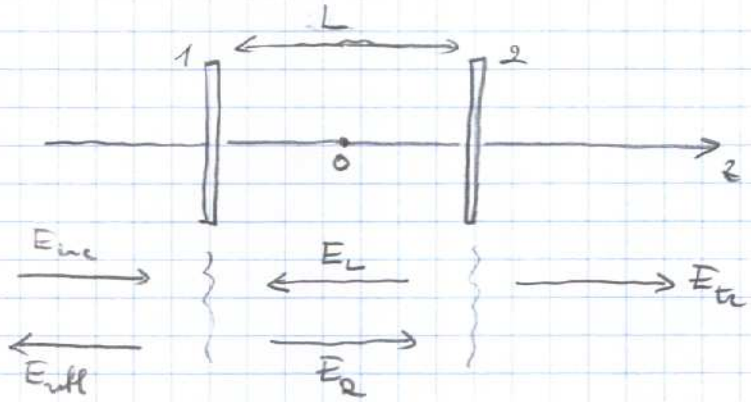
↳ more details in Jackson, "Classical E.D"

Phenomenology of cavities

simplest 1D model:

- light propagation along z
- single polarisation state
- mirrors at $z = \pm L/2$, characterised by transmittivity t and reflectivity r.

  $\hookrightarrow$ lossless condition: $|t|^2 + |r|^2 = 1$



mirror 2

$$E_{tr} = t \cdot E_R^{(2)}$$

$$E_L^{(2)} = r \cdot E_R^{(2)}$$

(1, 2) indicate spatial position of mirrors

mirror 1

$$E_{refl} = -r\, E_{inc} + t \cdot E_L^{(1)}$$

$$E_R^{(1)} = t \cdot E_{inc} + r \cdot E_L^{(1)}$$

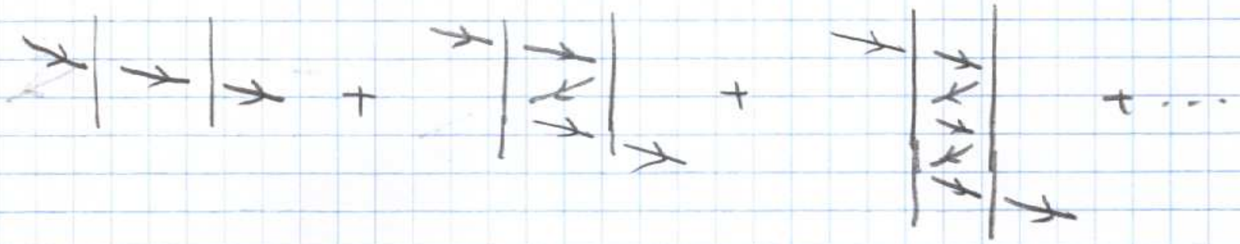
propagation between mirrors:

$$E_L^{(1)} = e^{ikL}\, E_L^{(2)}$$

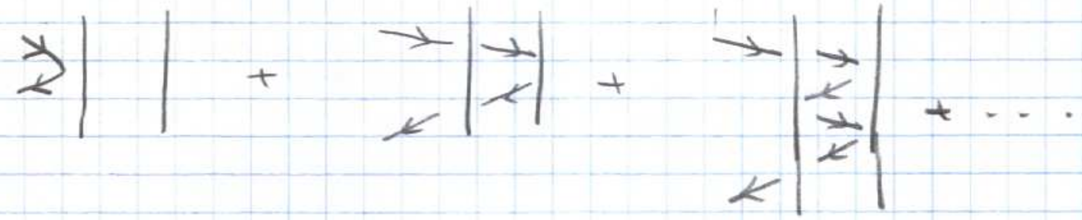
$$E_R^{(1)} = e^{-ikL}\, E_R^{(2)}$$

$$t_{tot} = \frac{E_{tr}}{E_{inc}} = \frac{t^2 e^{ikL}}{1 - r^2 e^{2ikL}} \simeq t^2 e^{ikL}\left[1 + r^2 e^{2ikL} + \ldots\right] =$$

$$= t^2 e^{ikL} + t^2 r^2 e^{3ikL} + t^2 r^4 e^{5ikL}$$



$$r_{tot} = -r\left[1 + \frac{t^2 e^{2ikL}}{1 - r^2 e^{2ikL}}\right] \simeq r + r t^2 e^{2ikL} + r^3 t^2 e^{4ikL} + \ldots$$



$$|t_{tot}|^2 = T_{tot} = \frac{T^2}{|1 - r^2 e^{2ikL}|^2}$$

most significant $|r|^2 = R = 1 - T \simeq 1$ case :

(i.e. $T \ll 1$)

* min $T_{tot} \simeq \frac{T^2}{2} \ll 1$

* max $T_{tot} = 1$ for $r^2 e^{2ikL} = R \in \mathbb{R}^+$

$\longrightarrow$ tend to quantized mode resonances for $R=1$, $T=0$
(isolated cavity with perfect mirrors)

* around maximum : $k = \bar{k} + \delta k$; $\bar{k} = N\Delta k$; $\Delta k = \pi/L$

$$T_{tot} = \frac{T^2}{\left|1 - R e^{2i\delta k L}\right|^2} \simeq$$

$$\simeq \frac{T^2}{\left[1 - R\left(1 - \frac{(2\delta k L)^2}{2}\right)\right]^2 + R^2 (2\delta k L)^2}$$

$$\simeq \frac{T^2}{T^2 + RT(2\delta k\, L)^2 + \dots + 4R^2 \delta k^2 L^2}$$

$$= \frac{1}{1 + 4\frac{R^2}{T^2} L^2 \delta k^2}$$

in frequency space : $\delta k = \delta\omega / c$ (assume cavity material has $n=1$):

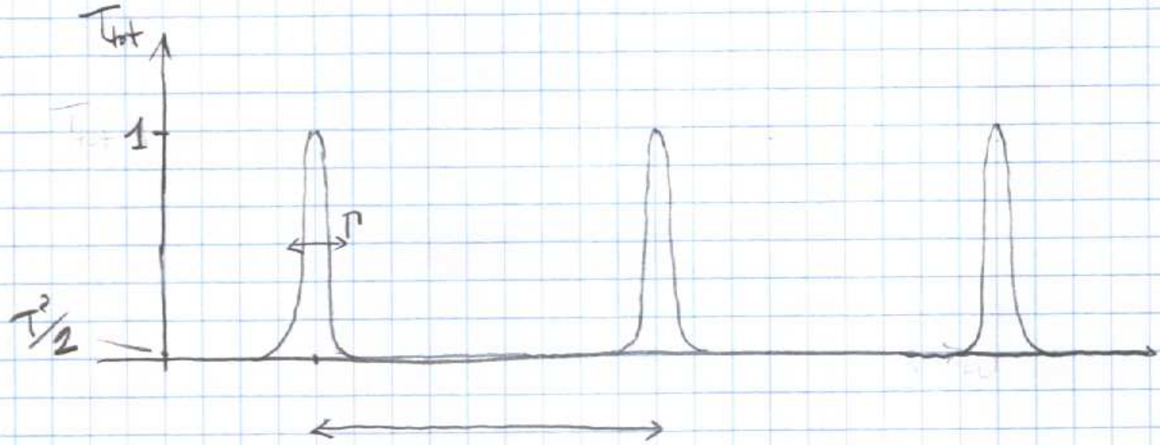
$$T_{tot}(\omega) = \frac{\frac{c^2 T^2}{4R^2L^2}}{\frac{c^2T^2}{4R^2L^2} + \delta\omega^2}$$

→ lorentzian shape with linewidth

$$\Gamma = \frac{cT}{RL} \simeq \frac{cT}{L}$$

Physical interpretation : every $\tau = \frac{L}{c}$ loses a fraction T of the loaded energy.

Q-factor of resonance = ω_0/Γ = # cycles before being damped out.



$\Delta\omega_{FSR} = {}^{c\pi}/_{L}$ free spectral range

$$\left[\!\left[\frac{\Delta\omega_{FSR}}{2\pi} = \Delta\nu_{FSR} = \frac{c}{2L} = \frac{1}{T_{round-trip}} \right]\!\right]$$

In-cavity field:

$$|E_R|^2 = \frac{1}{T}|E_{tr}|^2, \quad |E_L|^2 = \frac{R}{T}|E_{tr}|^2 \simeq \frac{1}{T}|E_{tr}|^2$$

$$\eta = \text{cavity enhancement} = \frac{|E_R|^2 \simeq |E_L|^2}{|E_{inc}|^2} = \frac{T_{tot}}{T}$$

↳ same series of Lorentzian peaks.

max height $= {}^{1}/_{T} \gg 1$, min $= {}^{T}/_{2} \ll 1$

* Energy penetrates into cavity only close to resonance
* Cavity enhancement by factor ${}^{1}/_{T} = {}^{\Gamma}/_{2\Delta\nu_{FSR}} = \frac{1}{2\pi}\mathcal{F}$
  ↳ Finesse of cavity
* Cavity mode weakly perturbed by losses $|E_R|^2 \simeq |E_L|^2$