# Lecture 3 : Radiation and scattering

## Oscillating dipole


$$\underline{x}(t) = \mathrm{Re}\left[\underline{x}_0\, e^{-i\omega_0 t}\right]$$

$$\underline{d}(\underline{r},t) \simeq q\, \underline{x}_0\, \delta(\underline{r})\, e^{-i\omega_0 t}$$

higher multipole moments $\sim q|x_0|^{n>1} = Q^{(n)}$

* dipole limit $q \to \infty$, $x_0 \to 0$ at fixed $q|x_0|$

$\hookrightarrow Q^{(n>1)} \to 0$.

Current $\quad J(\underline{r},t) = -i\omega_0 q\, \underline{x}_0\, \delta(\underline{r})\, e^{-i\omega_0 t}$

$J(\underline{k},t) = -i\omega_0 q\, \underline{x}_0\, e^{-i\omega_0 t}$ (independent of $\underline{k}$)

$$J_T(\underline{k},t) = -i\omega_0 q\, e^{-i\omega_0 t}\left[\underline{x}_0 - \frac{\underline{k}\,(\underline{k}\cdot\underline{x}_0)}{k^2}\right]$$

## Maxwell's equations

$$\left.\begin{aligned} i\underline{k} \times \underline{E}_T &= -\frac{1}{c}\frac{\partial \underline{B}_T}{\partial t} \\ i\underline{k} \times \underline{B}_T &= \frac{1}{c}\frac{\partial \underline{E}_T}{\partial t} + \frac{4\pi}{c} J_T \end{aligned}\right\} \text{eq. motion for transverse fields.}$$

longitudinal fields $\rightarrow$ follow instantaneously charge

$B_L = 0$, $\quad i\underline{k}\, \underline{E}_L = 4\pi\rho$

$$i\underline{k}\times(i\underline{k}\times B_T) = i\underline{k}\times\left(\frac{1}{c}\frac{\partial E_T}{\partial t} + \frac{4\pi}{c}J_T\right)$$

$$-\underline{k}\times(\underline{k}\times B_T) = \frac{1}{c}\left(i\underline{k}\times \dot{E}_T\right) + \frac{4\pi i}{c}\,\underline{k}\times J_T$$

$$k^2 B_T = -\frac{1}{c^2}\frac{\partial^2 B_T}{\partial t^2} + \frac{4\pi i}{c}\,\underline{k}\times \underline{J}_T$$

$$\boxed{\frac{\partial^2 B_T}{\partial t^2} = -c^2k^2\,\underline{B}_T + 4\pi i c\,(\underline{k}\times J_T)}$$

* every $\underline{k}$-mode is a driven harmonic oscillator
* resonance frequency $c\cdot|k|$
* linewidth $= 0$

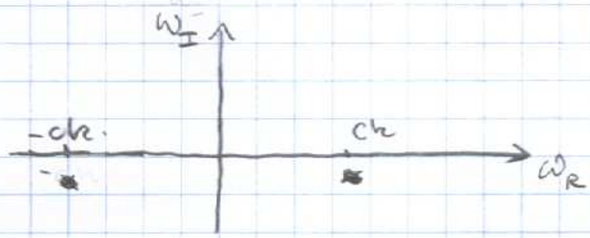
<u>To regularise integrals</u> :

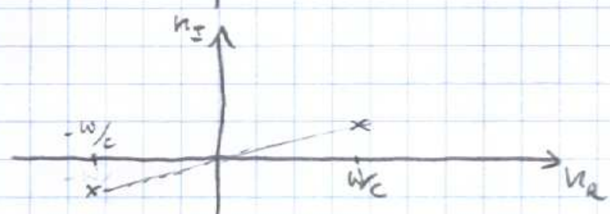
$$\frac{\partial^2 B_T}{\partial t^2} = -c^2k^2 B_T - \gamma\frac{\partial B_T}{\partial t} + 4\pi i c(\underline{k}\times J_T)$$

with $\gamma \to 0^+$ : sort of friction.

implements <u>causal</u> response

$\longrightarrow$ resonance <u>poles</u> at $\omega^2 + i\omega\gamma - c^2k^2 = 0$.



Stationary - state (oscillating at $\omega_0$)

$$\left(c^2 k^2 - \omega_0^2 - i\omega_0\gamma\right) B_T(\underline{k}) =$$

$$= 4\pi i c \left[ \underline{k} \times \left(-i\omega_0 q\right) \left[ \underline{x}_0 - \cancel{\frac{\underline{k}\cdot(\underline{k}\cdot\underline{x}_0)}{k^2}} \right]\right)\right]$$

$(\underline{k} \times \underline{k} = 0)$

$$= 4\pi c \omega_0 q \left(\underline{k} \times \underline{x}_0\right)$$

$$\underline{B}(\underline{r}) = \int \frac{d^3 \underline{k}}{(2\pi)^3} e^{i\underline{k}\cdot\underline{r}} \left[ B_T(\underline{k}) + \cancel{B_L(\underline{k})}^{\,0} \right] =$$

$$= \int \frac{d^3 \underline{k}}{(2\pi)^3} e^{i\underline{k}\cdot\underline{r}}\, 4\pi c q \omega_0 \left(\underline{k} \times \underline{x}_0\right) \frac{1}{c^2 k^2 - \omega_0^2 - i\omega_0\gamma}$$

* resonant denominator
* regularisation of pole required

Take $\underline{r}$ along $\hat{z}$ axis

Split $\underline{x}_0$ into $\underline{x}_0 = x_{\parallel}\,\hat{z} + x_{\perp}\cdot\hat{x}$

orthogonal

parallel to line of sight

$$B(z) = \frac{1}{(2\pi)^3} \int k^2 dk \, d^2\Omega \; e^{i k_z z} \, 4\pi q \omega_0 c \, \left( \underline{k} \times \left[ x_\parallel \hat{z} + x_\perp \hat{x} \right] \right) \cdot \frac{1}{c^2 k^2 - \omega_0^2 - i \omega_0 \gamma}$$

solid angle

$\int d^2\Omega \, (\underline{k} \times \hat{z}) \, e^{i k_z z} = 0$ as integrand is odd under $(x, y, z) \longrightarrow (-x, -y, z)$

$$\underline{k} \times \hat{x} = (k_x \hat{x} + k_y \hat{y} + k_z \hat{z}) \times \hat{x} =$$

$$= -k_y \hat{z} + k_z \hat{y}$$

$\int d^2\Omega \; k_y \hat{z} \; e^{i k_z z} = 0$ (same reason)

$$B(z) = \frac{1}{(2\pi)^3} \int dk \, k^2 \int d^2\Omega \; e^{i k_z z} \, 4\pi q \omega_0 c \, k_z \cdot x_\perp \hat{y} \, \frac{1}{c^2 k^2 - \omega_0^2 - i \omega_0 \gamma}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty dk \, k^2 \int_{-1}^{1} d\cos\vartheta \; e^{i k z \cos\vartheta} \, 4\pi q \omega_0 c \, x_\perp \, \hat{y} \, \frac{k \cos\vartheta}{c^2 k^2 - \omega_0^2 - i \omega \gamma}$$

$$= \frac{c q \omega_0 x_\perp}{2\pi} \, \hat{y} \int_{-\infty}^{\infty} dk \, \frac{k^3}{c^2 k^2 - \omega_0^2 - i \omega_0 \gamma} \cdot \int_{-1}^{1} d\cos\vartheta \, \cos\vartheta \, e^{i k z \cos\vartheta}$$

extends integral: $(k \to -k, \ \cos\vartheta \to -\cos\vartheta)$

$$\int_{-1}^{1} dx \cdot x \; e^{i\alpha x} = e^{i\alpha x}\left[\frac{x}{i\alpha} - \frac{1}{(i\alpha)^2}\right]_{-1}^{1} =$$

$$= e^{i\alpha x}\left[\frac{1}{\alpha^2} - i\frac{x}{\alpha}\right]_{-1}^{1} = e^{i\alpha x}\left[\frac{1}{\alpha^2} - \frac{ix}{\alpha}\right] - c.c$$

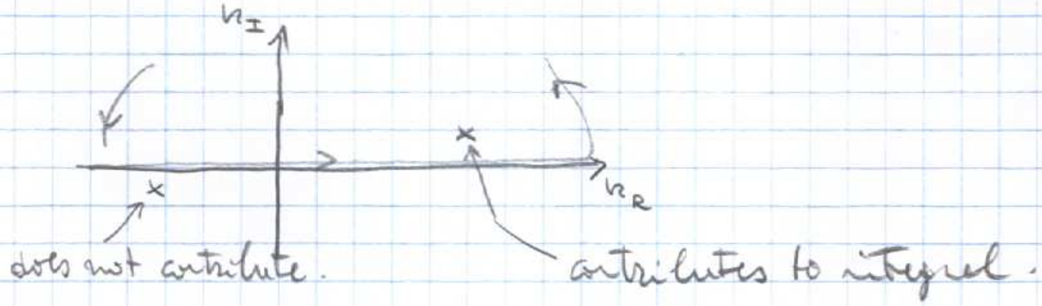
$$B(z) = \frac{cq\omega_0 x_\perp}{2\pi} \cdot \hat{y} \int_{-\infty}^{\infty} dk \, \frac{k^3}{c^2k^2 - \omega_0^2 - i\omega_0\gamma} \cdot$$

$$\cdot \left( e^{ikz}\left[\frac{1}{(kz)^2} - \frac{i}{kz}\right] - c.c. \right)$$

$$= \frac{cq\omega_0 x_\perp}{2\pi} \hat{y} \int_{-\infty}^{\infty} dk \left( e^{ikz}(1 - ikz) - c.c. \right) \frac{k/z^2}{c^2k^2 - \omega_0^2 - i\omega_0\gamma}$$

* $e^{ikz}$ integral can be closed in upper half-plane.

$$e^{ikz} = e^{ik_R z} \, e^{-k_I z} \longrightarrow 0 \quad \text{for } k_I \to \infty .$$




* $e^{-ikz}$ integral $\longrightarrow$ lower half-plane, poles exchanged.

$$= \frac{q\omega_0 x_\perp}{2\pi c r^2}\hat{y}\int_{-\infty}^{\infty} dk \left[e^{ikr} k(1-ikr) - e^{-ikr} k(1+ikr)\right] \cdot \frac{1}{k^2-\omega_0^2/c^2 - i\omega_0\gamma/c^2}$$

ACW loop, pole at $\omega_0/c + i0^+$

$$= \frac{q\omega_0 x_\perp}{2\pi c r^2}\hat{y}\left[2\pi i \cdot \cancel{\frac{\omega_0}{c}}\left(1-i\frac{\omega_0}{c}r\right)\cdot \frac{e^{i\frac{\omega_0}{c}r}}{2\cancel{\omega_0/c}} + \right.$$

$$\left. - (-2\pi i)\left(-\cancel{\frac{\omega_0}{c}}\right)\left(1-i\left(-\frac{\omega_0}{c}\right)r\right)\cdot\frac{e^{-i\left(-\frac{\omega_0}{c}\right)r}}{-2\cancel{\omega_0/c}}\right]$$

CW loop, pole at $-\omega_0/c - i0^+$

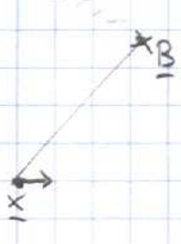
$$= \frac{q\omega_0 x_\perp}{\cancel{2\pi} c r^2}\hat{y}\cdot\cancel{2\pi} i\cdot\cancel{2}\left(1-i\frac{\omega_0}{c}r\right)\frac{e^{i\omega_0 r/c}}{\cancel{2}} =$$

$$= i\frac{q\omega_0 x_\perp}{c}\hat{y}\left(1-i\frac{\omega_0}{c}r\right)\frac{e^{i\omega_0 r/c}}{r^2}$$

$$= q x_\perp \hat{y}\left(\frac{\omega_0}{c}\right)^2\frac{e^{i\omega_0 r/c}}{r}\left(\frac{i}{\frac{\omega_0}{c}r}+1\right) = \underline{B}(\underline{r})$$

<u>in vector form</u>:

$$\underline{B}(\underline{r}) = \left[\hat{n}\times(q\,\underline{x})\right]\cdot\frac{\omega_0^2}{c^2}\frac{e^{i\frac{\omega_0}{c}r}}{r}\left(1+\frac{i}{\frac{\omega_0}{c}r}\right)\Bigg]$$




$\hat{n}$ = unit vector along $\underline{r}$

→ dominant radiation field $\sim 1/r$ at large $r\frac{\omega_0}{c} \gg 1$

Electric field $\longrightarrow$ transverse + longitudinal

transverse $\rightarrow$ propagating modes

longitudinal $\rightarrow$ static dipole field

$$\underline{E}_L(\underline{r}) = \frac{3\hat{n}\left[\hat{n}\cdot(q\underline{x})\right] - q\underline{x}}{r^3}$$

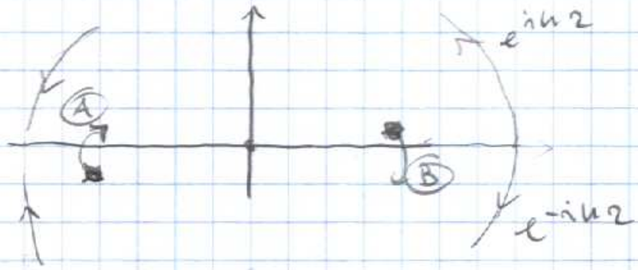
$$\underline{E}_T(\underline{r}) = \ldots$$

performing all the integrals again:

$$\underline{E}(\underline{r}) = \underbrace{k^2\left[\hat{n}\times(q\underline{x})\right]\times\hat{n}\,\frac{e^{i\frac{\omega_0}{c}r}}{r}}_{\text{radiation field}} + \underbrace{\left[3\left(\hat{n}\cdot(q\underline{x})\right)\hat{n} - (q\underline{x})\right]\left(\frac{1}{r^3} - \frac{i\omega_0}{r^2 c}\right)e^{i\frac{\omega_0}{c}r}}_{\text{electrostatic component}}$$

## Remarks on the poles

moving




A $\begin{cases} \rightarrow \text{adds in-going } e^{-i\frac{\omega_0}{c}r} \text{ contribution to } e^{ikr} \text{ integral} \\ \rightarrow \text{removes out-going } e^{i\frac{\omega_0}{c}r} \text{ from } e^{-ikr} \text{ integral} \end{cases}$

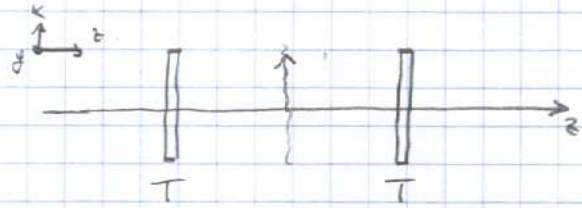
B analogous result

→ A + B transforms emission of radiation into "absorption" of in-going wave by dipole.

# Emission into a cavity: Purcell enhancement

simplified model: 1D cavity + plane layer of emitters.

$$\underline{J}(r,t) = \hat{e}_x\, J_0\, \delta(z)\, e^{-i\omega_0 t}$$



$$\nabla\times B = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c} J$$

$$\nabla\times\left(-\frac{1}{c}\frac{\partial B}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \frac{4\pi}{c^2}\frac{\partial J}{\partial t}$$

$$\nabla\times(\nabla\times E) = -\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \frac{4\pi}{c^2}\frac{\partial J}{\partial t}$$

only transverse $E$

$$+\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial J}{\partial t}$$

propagation along $z$.

$$\partial_z^2 E_x - \frac{1}{c^2}\partial_t^2 E_x = \frac{4\pi}{c^2}\partial_t J_x$$

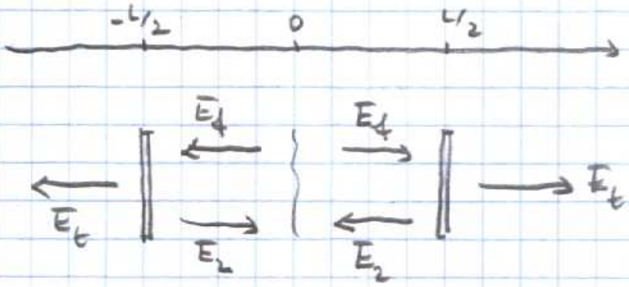
$$\Downarrow$$

$$\partial_z^2 E_x + \frac{\omega_0^2}{c^2} E_x = -\frac{4\pi i\omega_0}{c^2} J_x$$

$$\int_{0^-}^{0^+} dz\, \partial_z^2 E_x + \frac{\omega_0^2}{c^2} E_x = -\frac{4\pi i\omega_0}{c^2}\int_{0^-}^{0^+} dz\, J_x$$

$$\partial_z E_x\Big|_{0^-}^{0^+} = -\frac{4\pi i\omega_0}{c^2} J_0$$

boundary condition at emitting dipole layer.




dipoles at cavity center $\longrightarrow$ symmetry $z \to -z$

$$\left.\begin{aligned} z>0)\quad & E_x(z) = E_f\, e^{ikz} + E_2\, e^{-ikz} \\ z<0)\quad & E_x(z) = E_f\, e^{-ikz} + E_2\, e^{ikz} \end{aligned}\right\}\quad |z|<L/2$$

$$\partial_z E_x\Big|_{0^-}^{0^+} = ik\,(E_f - E_2) - \left[-ik\,(E_f - E_2)\right] =$$
$$= 2ik\,(E_f - E_2)$$

$$2ik\,(E_f - E_2) = -\frac{4\pi i \omega_0}{c^2} J_0$$

reflection at mirror: $E_2\, e^{-ikL/2} = r\, E_f\, e^{ikL/2}$

$$\cancel{2}\cancel{i}k\, E_f\,(1 - r\, e^{ikL}) = -\frac{\overset{2}{\cancel{4}}\pi \cancel{i} \omega_0}{c^2} J_0$$

$$E_f = -\frac{2\pi J_0}{c}\,\frac{1}{1 - r\, e^{ikL}}$$

$$\boxed{\underline{\text{Emitted field}}:\quad E_t = -\frac{2\pi J_0 t}{c}\,\frac{1}{1 - r\, e^{ikL}}}$$

free space emission:

$$E_t = -2\pi \frac{J_0}{c}$$

resonant cavity enhancement : $kL = 2\pi N$ , $1-r^2 = t^2 \ll 1$

$$E_t = -2\pi \frac{J_0 t}{c} \frac{1}{1-r} = -2\pi \frac{J_0}{c} \frac{t(1+r)}{\underbrace{1-r^2}_{t^2}} = -2\pi \frac{J_0}{c} \frac{2}{t}$$

off-resonant suppression :

$$E_t = -2\pi \frac{J_0 t}{c} \cdot \frac{1}{1+r} \simeq -2\pi \frac{J_0}{c} \cdot \frac{t}{2}$$

$$\frac{I_{free}}{I_{off\,res}} = \frac{I_{res}}{I_{free}} = 1/T \gg 1$$

$\rightarrow$ Purcell factor

Lineshape :

$$E_t \simeq -2\pi \frac{J_0 t}{c} \frac{1}{1-r-ir(kL-2\pi N)+\dots} \simeq$$

$$\simeq -2\pi \frac{J_0 t}{c} \frac{1}{1-r-iL\,\Delta\omega/c} = -2\pi i \frac{J_0 t}{c} \cdot \frac{c}{L} \frac{1}{\Delta\omega + i(1-r)c/L}$$

$$= -2\pi i \frac{J_0 t}{L} \cdot \frac{1}{\Delta\omega + i\frac{Tc}{2L}} = -2\pi i \frac{J_0 t}{L} \cdot \frac{1}{\Delta\omega + i\Gamma/2} = E_t$$

* linewidth = cavity mode linewidth $\Gamma$.

* emission amplitude $\sim 1/L$

<u>Radiative reaction and energy balance</u>

free-space emission of dipole sheet.

$$E(z) = -2\pi \frac{J_0}{c} e^{i\frac{\omega}{c}|z|}$$

work exerted by e.m. field on dipoles:

$$\mathcal{L} = \Delta d \cdot E(0)$$

electric field

work/unit surface

variation of dipole moment

* B field does not exert any work ($F = q\, \underline{v} \times \underline{B}$ is orthogonal to displacement $\underline{v}\,\Delta t$)

$$\Delta d = J_0 e^{-i\omega_0 t} \cdot \Delta t$$

<u>Power</u>: $W = \frac{1}{2} \mathrm{Re}\left[J_0 \cdot E^*(0)\right] = -\pi \frac{|J_0|^2}{c}$ (per unit surface)

E.m. field absorbs energy from current.

Energy emitted into radiation

↳ Poynting vector : $S = \frac{1}{8\pi} \mathrm{Re}\left[E \times B^*\right]$

↳ Emission on two sides.

$$S_{Tot.} = 2 \times \frac{1}{8\pi} \mathrm{Re}\left[|E_t|^2\right] = \frac{1}{4\pi} \left|\frac{2\pi J_0}{c}\right|^2 = \pi \frac{|J_0|^2}{c}$$

which coincides with power exerted on e.m. field.

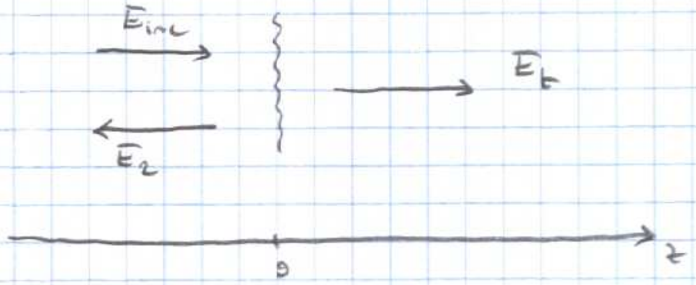
* Emitted field exerts force that opposes dipole motion

↳ radiative reaction

→ radiative linewidth

↓ "Lamb" shift

## Scattering on dipole sheet




example: scattering on quantum-well exciton

$$\partial_z E_x \Big|_{0^-}^{0^+} = ik E_t - \left( ik E_{inc} - ik E_r \right) =$$

$$= i\frac{\omega}{c}\left[ E_t - E_{inc} + E_r \right]$$

$$\partial_z E_x \Big|_{0^-}^{0^+} = -\frac{4\pi i \omega}{c^2} J_0 = -\frac{4\pi i \omega}{c^2} \chi_0(\omega)\, E_x(z=0)\, (-i\omega)$$

$$= -\frac{4\pi\omega^2}{c^2} \chi_0(\omega) \cdot E_t$$

resonant response: $\chi_0(\omega) = \dfrac{f}{\omega_0^2 - \omega^2 - i\omega\gamma}$

non-radiative broadening

$$i\frac{\cancel{\omega}}{c}\left(E_t - E_{inc} + E_r\right) = -\frac{4\pi\omega^{\cancel{2}}}{c^{\cancel{2}}}\frac{f}{\omega_0^2-\omega^2-i\omega\gamma} \quad E_t$$

E-field continuity: $E_{inc} + E_r = E_t$

$$2E_r = \frac{4\pi f\omega i}{c}\frac{1}{\omega_0^2-\omega^2-i\gamma\omega}\left(E_{inc}+E_r\right)$$

$$E_2 \left[ 1 - \frac{2\pi i f \omega}{c} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \right] = - \frac{2\pi f i}{c} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \quad E_{inc}$$

$$r = \frac{E_2}{E_{inc}} = \frac{- \frac{2\pi i f \omega}{c} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}}{1 - \frac{2\pi i f \omega}{c} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}} =$$

$$= \frac{- 2\pi i f \omega / c}{\omega_0^2 - \omega^2 - i\gamma\omega - i\, 2\pi f \omega / c} \longrightarrow \text{Lorentzian shape}$$

Purely radiative limit $\gamma \to 0^+$ :

* $r(\omega = \omega_0) = 1$
* linewidth $\Gamma_{rad} = 2\pi f / c$ : RADIATIVE.
* no Lamb-shift : peculiar to 1D geometry.

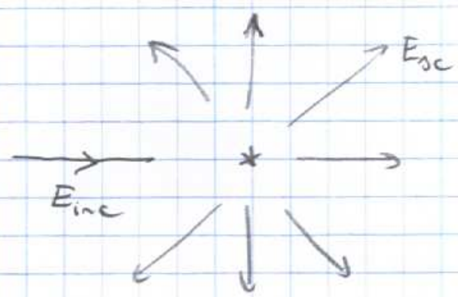
General case :

* $r(\omega = \omega_0) = \dfrac{2\pi f / c}{\gamma + 2\pi f / c}$

"Isolated atom" : localised harmonic oscillator

* 3D continuum of modes.

* spherical scattered wave (+ polarization dependence)




$$E_{sc}(\omega) = \frac{\alpha}{\omega_0^2 - \omega^2 - i\gamma\omega - i\omega\Gamma_{rad} - \omega\Delta_{rad}}$$

also frequency shift may be present.

* Calculation made harder by UV divergence problems.
  + frequency shift due to longitudinal modes