Lecture 7 : quantization of the e.m. field

longitudinal field : instantaneously follows charges.

$\nabla \cdot B = 0, \qquad \nabla \cdot E = 4\pi\rho$

charges create

Transverse field : collection of harmonic oscillators driven by currents.

$$\frac{\partial^2 B_T}{\partial t^2} = -c^2 k^2 B_T + 4\pi i c \left(\underline{k} \times \underline{J}_T \right)$$

$$i\underline{k} \times \underline{E}_T = -\frac{1}{c}\frac{\partial B_T}{\partial t}$$

both fields then act back onto charges

$$F = q \cdot \underline{E} + \frac{q}{c}\, \underline{v} \times \underline{B}$$

Degrees of freedom :

- charge motion
- transverse e.m. field :
  collection of harmonic oscillators
  1 h.o. per k vector in reciprocal space.

Quantization

- naif approach : replace h.o. variables with operators
- Lagrangian / Hamiltonian approach

Reference book:

C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg
"Introduction to QED",

Standard Lagrangian of E.D.:

$$\mathcal{L} = \sum_\alpha \frac{1}{2} m \dot{\underline{r}}_\alpha^2 + \frac{1}{8\pi} \int d^3r \left[\underline{E}(r)^2 - \underline{B}(r)^2\right]$$

$$+ \sum_\alpha \left[\frac{q_\alpha}{c} \dot{\underline{r}}_\alpha \cdot \underline{A}(r_\alpha) - q_\alpha U(r_\alpha)\right] =$$

$$= \mathcal{L}\left(r_\alpha, \dot{r}_\alpha ; A(r), \dot{A}(r), U(r), \dot{U}(r)\right)$$

in $\mathcal{L}$: $\quad E(r) = -\nabla U(r) - \frac{1}{c}\dot{A}(r)$

$$B(r) = \nabla \times A(r)$$

interaction term can be rewritten as:

$$\int d^3r \left[\frac{1}{c} j(r) A(r) - \rho(r) \cdot U(r)\right]$$

Covariant form of $\mathcal{L}$:

$$\mathcal{L} = -\frac{c^2}{16\pi} F_{\mu\nu} F^{\mu\nu} - \sum_\alpha q_\alpha \, dx_\alpha^\mu A_\mu - \sum_\alpha m_\alpha c^2 \, d\tau_\alpha$$

in Fourier space

$$\mathcal{L} = \frac{1}{2} m_\alpha \dot{r}_\alpha^{\,2} + \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{4\pi} \left[|\tilde{E}(k)|^2 - |\tilde{B}(k)|^2 \right] + \right.$$

$$+ \frac{1}{c}\left(\tilde{j}^*(k) \cdot \tilde{A}(k) + \tilde{A}^*(k) \cdot \tilde{j}(k) \right) - \tilde{\rho}^*(k)\, \tilde{U}(k) +$$

$$\left. - \tilde{U}^*(k)\, \rho(k) \right\}$$

NOTE:

- $\tilde{A}(k)$, $\tilde{U}(k)$ are complex variables.
- reality of $A(r)$, $U(r)$ $\Longrightarrow$ $\tilde{A}(-k) = \tilde{A}^*(k)$; $U(-k) = U^*(k)$
- fields in half reciprocal space are independent
  $\hookrightarrow$ integral $\int$ in $\mathcal{L}$ limited to half-space.
- real and imaginary parts of $A(k)$, $U(k)$ are independent real variables

A parenthesis: complex coordinates in lagrangian

$\mathcal{L}(x_1, x_2\,;\, \dot{x}_1, \dot{x}_2)$ generic lagrangian (real)

change of variables $(x_1, x_2) \longrightarrow (X, X^*)$

$$X = \frac{1}{\sqrt{2}}(x_1 + i x_2), \quad X^* = \frac{1}{\sqrt{2}}(x_1 - i x_2)$$

generic $f(x_1, x_2)$ $\qquad df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2$

in complex coordinates:

$$df = \frac{\partial f}{\partial X}\, dX + \frac{\partial f}{\partial X^*}\, dX^*$$

with
$$\begin{cases} \dfrac{\partial}{\partial X} = \dfrac{1}{\sqrt{2}}\left(\dfrac{\partial}{\partial x_1} - i\,\dfrac{\partial}{\partial x_2}\right) \\[2ex] \dfrac{\partial}{\partial X^*} = \dfrac{1}{\sqrt{2}}\left(\dfrac{\partial}{\partial x_1} + i\,\dfrac{\partial}{\partial x_2}\right) \end{cases}$$

indeed
$$\frac{\partial}{\partial X} X = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right)\frac{(x_1 + i x_2)}{\sqrt{2}} = 1$$

$$\frac{\partial}{\partial X} X^* = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_1}\right)\frac{(x_1 - i x_2)}{\sqrt{2}} = 0.$$

Conjugate momentum:

$$P = \frac{\partial \mathcal{L}}{\partial \dot{X}^*} = \frac{1}{\sqrt{2}}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1} + i\,\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right) = \frac{1}{\sqrt{2}}\left(p_1 + i\,p_2\right)$$

Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}^*} - \frac{\partial \mathcal{L}}{\partial X^*} = 0 \implies \frac{d}{dt}\frac{1}{\sqrt{2}}(p_1 + i p_2) - \frac{1}{\sqrt{2}}\left(\frac{\partial \mathcal{L}}{\partial x_1} + i\,\frac{\partial \mathcal{L}}{\partial x_2}\right) = 0$$

separating real and imaginary parts: $\dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{x}_{1,2}} - \dfrac{\partial \mathcal{L}}{\partial x_{1,2}} = 0$

Hamiltonian

$$H = \dot{x}_1 p_1 + \dot{x}_2 p_2 - \mathcal{L} = \dot{X} P^* + \dot{X}^* P - \mathcal{L}$$

Commutators

$$[X, P] = \left[\frac{1}{\sqrt{2}}(x_1 + i x_2),\ \frac{1}{\sqrt{2}}(p_1 + i p_2)\right] = 0$$

$$[X, P^*] = i\hbar$$

Back to E.D.

$$\mathcal{L} = \sum_\alpha \frac{1}{2} m_\alpha \dot{r}_\alpha^2 + \int \frac{d^3k}{(2\pi)^3} \cdot \left\{ \frac{1}{4\pi} \left[\left| \frac{\dot{\tilde{A}}(k)}{c} + i k \tilde{U}(k) \right|^2 + \right.\right.$$

$$\left. - \left| k \times \tilde{A}(k) \right|^2 \right] + \frac{1}{c}\left(j^*(k)\, A(k) + j(k) \cdot A^*(k)\right) +$$

$$\left. - \rho(k)\, U^*(k) - \rho^*(k)\, U(k) \right] =$$

$$= \sum_\alpha \frac{1}{2} m_\alpha \dot{r}_\alpha^2 + \frac{q_\alpha}{c} \cdot \dot{r}_\alpha \cdot A(r_\alpha) - q_\alpha U(r_\alpha) +$$

$$+ \int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{4\pi} \left[\left| \frac{\dot{A}(k)}{c} + i k\, \tilde{U}(k) \right|^2 - \left| k \times A(k) \right|^2 \right]$$

eq. motion for particles

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}_\alpha} - \frac{\partial \mathcal{L}}{\partial r_\alpha} = 0$$

$$\frac{d}{dt}\left(m \dot{r}_\alpha + \frac{q_\alpha}{c} A(r_\alpha)\right)_i - \frac{q_\alpha}{c} \dot{r}_{\alpha,j} \cdot \nabla_i A_j + q_\alpha \nabla_i U = 0$$

$$m \ddot{r}_{\alpha,i} = - q_\alpha \nabla_i U - \frac{q_\alpha}{c} \left[\underbrace{\frac{\partial A}{\partial t} + \dot{r}_{\alpha,j} \nabla_j A_i}_{\frac{d}{dt} A(r_\alpha)} - \dot{r}_{\alpha,j} (\nabla_i A_j) \right]$$

$$m\,\ddot{r}_{\alpha,j} = q_\alpha\left[-\nabla_i U - \frac{1}{c}\frac{\partial A}{\partial t}\right] + \frac{q_\alpha}{c}\,\dot{r}_\alpha \times (\nabla\times A)\Big|_i$$

↳ Lorentz force

in fact:

$$\dot{r}_\alpha \times (\nabla\times A)\Big|_i = \varepsilon_{ijk}\,\dot{r}_{\alpha,j}\,\varepsilon_{klm}\frac{\partial}{\partial x_l}A_m =$$

$$= \varepsilon_{kij}\,\varepsilon_{klm}\,\dot{r}_{\alpha j}\frac{\partial}{\partial x_l}A_m =$$

$$= \dot{r}_{\alpha j}\left(\frac{\partial}{\partial x_i}A_j\right) - \dot{r}_{\alpha j}\frac{\partial}{\partial x_j}A_i$$

eq. motion for field: i) scalar potential U

$$\frac{\partial\mathcal{L}}{\partial U^*} = -\frac{1}{4\pi}\,i\underline{k}\cdot\left(\frac{\dot{\underline{A}}}{c} + i\underline{k}\,U\right) - \rho$$

$$\frac{\partial\mathcal{L}}{\partial\dot{U}^*} = 0$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{U}^*} - \frac{\partial\mathcal{L}}{\partial U} = 0 \qquad \text{fixes instantaneous value of } U$$

$$4\pi\rho = i\underline{k}\cdot\left(-\frac{1}{c}\dot{A} - i\underline{k}\,U\right)$$

i.e. $i\underline{k}\,\underline{E} = 4\pi\rho$ (Coulomb eq.)

ii) vector potential A

$$\frac{\partial\mathcal{L}}{\partial A^*} = \frac{j(k)}{c} + \frac{1}{4\pi}\,\underline{k}\times(\underline{k}\times\underline{A})$$

$$\frac{\partial\mathcal{L}}{\partial\dot{A}^*} = \frac{1}{4\pi c^2}\left(\dot{\underline{A}} + i\underline{k}\cdot U\right)$$

$$\frac{d}{dt}\left(\frac{1}{4\pi c}\left(\frac{\mathring{\underline{A}}}{c}+ik\,U\right)\right) = \frac{j(k)}{c} + \frac{1}{4\pi}\,\underline{k}\times(\underline{k}\times\underline{A})$$

$$\frac{1}{c}\frac{d}{dt}\underbrace{\left(-\frac{1}{c}\mathring{A} - ik\,U\right)}_{E} = -\frac{4\pi}{c}\,j(k) + ik\times(\underbrace{ik\times\underline{A}}_{B})$$

$$ik\times B = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}j \longrightarrow \text{Maxwell eq}$$

The two other Maxwell's eqs:

$\nabla\cdot B = 0$ automatic from $B = \nabla\times A$

$\nabla\times E = -\frac{1}{c}\frac{\partial B}{\partial t}$ automatic from $E = -\nabla U - \frac{1}{c}\frac{\partial A}{\partial t}$

in fact $\nabla\times\left(-\nabla U - \frac{1}{c}\frac{\partial A}{\partial t}\right) = -\cancel{\nabla\times(\nabla U)}^{\,0} - \frac{1}{c}\nabla\times\frac{\partial A}{\partial t} =$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\nabla\times A = -\frac{1}{c}\frac{\partial B}{\partial t}$$

Simplify the Lagrangian

* U is not a real degree of freedom, as it is slaved to A and particle position. Replacing in $\mathcal{L}$ its expression $U = \frac{1}{k^2}\left(ik\frac{\dot{A}_{\parallel}}{c} + 4\pi\rho\right)$ gives:

$$\mathcal{L} = \sum_\alpha \frac{1}{2}m_\alpha \dot{r}_\alpha^2 - \int\frac{d^3k}{(2\pi)^3}\frac{4\pi}{k^2}|\rho(k)|^2 + \frac{1}{4\pi}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{c^2}|\mathring{A}_\perp|^2 - (k\times A_\perp^*)(k\times A_\perp)\right]$$

$$+\int\frac{d^3k}{(2\pi)^3}\left(j^*A_\perp + A_\perp^* j\right) + \frac{d}{dt}\int\frac{d^3k}{(2\pi)^3}\frac{i}{k}\left[\rho A_\parallel^* - \rho^* A_\parallel\right]$$

* $A_\parallel$ only appears in total derivative term that can be eliminated

$$\Rightarrow \mathcal{L}(r_\alpha, \dot{r}_\alpha ; A_\perp, A_\perp^* ; \dot{A}_\perp, \dot{A}_\perp^*)$$

* Coulomb term can be rewritten as

$$-\frac{1}{2}\sum_{\alpha,\alpha'} \frac{q_\alpha q_{\alpha'}}{|r_\alpha - r_{\alpha'}|} = -\int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{k^2} |\rho(k)|^2 + \text{self energy of particles}$$

In real space:

$$\mathcal{L} = \sum_\alpha \frac{1}{2} m \dot{r}_\alpha^2 - \frac{1}{2}\sum_{\alpha\alpha'} \frac{q_\alpha q_{\alpha'}}{|r_\alpha - r_{\alpha'}|} + \frac{1}{8\pi}\int d^3r \left[\frac{1}{c^2}\dot{A}_\perp^2 - (\nabla\times A_\perp)^2\right] +$$

$$+ \sum_\alpha \frac{q_\alpha}{c} \dot{r}_\alpha \cdot A_\perp(r_\alpha) = \mathcal{L}(r_\alpha, \dot{r}_\alpha, A_\perp, \dot{A}_\perp)$$

* longitudinal field $\longrightarrow$ Coulomb energy

* Transverse field $\longrightarrow$ radiative effects

NOTE: even if the "COULOMB GAUGE" $\nabla\cdot A = 0$ $(A_L = 0)$ is not a covariant concept, the physics of the e.m. field is described exactly. If needed, a relativistic description of particles can be included

Conjugate momenta

$$\Pi(k) = \frac{\partial \mathcal{L}}{\partial \dot{A}(k)^*} = \frac{1}{4\pi c^2} \dot{A}(k)$$

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{r}_\alpha} = m \dot{r}_\alpha + \frac{q_\alpha}{c} A(r_\alpha)$$

$$H = \int \frac{d^3k}{(2\pi)^3} \left[\Pi^*(k) \dot{A}(k) + \Pi(k) \dot{A}^*(k) + p_\alpha \dot{r}_\alpha \right] - \mathcal{L} =$$

$$= \sum_\alpha \frac{1}{2m_\alpha} \left(p_\alpha - \frac{q_\alpha}{c} A_\alpha(r_\alpha) \right)^2 + \frac{1}{2} \sum_{\alpha\alpha'} \frac{q_\alpha q_{\alpha'}}{|r_\alpha - r_{\alpha'}|} +$$

$$+ \frac{1}{4\pi} \int \frac{d^3k}{(2\pi)^3} \left[c^2 |4\pi \Pi(k)|^2 + k^2 |A(k)|^2 \right]$$

* light-matter interaction via minimal coupling replacement $P \to P - \frac{q}{c} A$

* field evolution $\longrightarrow$ quadratic H in X, P $\Longrightarrow$ collection of h.o.

* for every k in half-space
  - $\to$ 2x polarization states
  - $\to$ real + imaginary part of A, Π

Upon quantization:

$$[A(k), \Pi(k')] = 0 \quad , \quad [A(k), \Pi(k')^*] = i\hbar (2\pi)^3 \delta^3(k-k')$$

Define

for $k \in$ H.S.
$$\begin{cases} \alpha(k) = \sqrt{\dfrac{1}{8\pi\hbar c k}}\left[k\, A(k) + 4\pi i c\, \Pi(k)\right] \\ \\ \alpha^{\dagger}(k) = \sqrt{\dfrac{1}{8\pi\hbar c k}}\left[k\, A^{\dagger}(k) - 4\pi i c\, \Pi^{\dagger}(k)\right] \end{cases}$$

to obtain a full basis, has to introduce also:

$$\beta(k) = \sqrt{\frac{1}{8\pi\hbar c k}}\left[k\, A^{\dagger}(k) + 4\pi i c\, \Pi^{\dagger}(k)\right]$$

and $$\beta^{\dagger}(k) = \sqrt{\frac{1}{8\pi\hbar c k}}\left[k\, A(k) - 4\pi i c\, \Pi(k)\right]$$

NOTE: $\beta, \beta^{\dagger}$ are linearly independent from $\alpha, \alpha^{\dagger}$.

together form a basis of the operator space $A(k), A^{\dagger}(k), \Pi(k), \Pi^{\dagger}(k)$
are needed to form field observables,

$$A_R(k) = A(k) + A^{\dagger}(k) \quad \ldots$$

Commutation rules:

$$\left[\alpha(k), \alpha(k')\right] = 0$$

$$\left[\alpha(k), \alpha^{\dagger}(k')\right] = \frac{1}{8\pi\hbar c k}\left[-4\pi i c k\,(i\hbar) + 4\pi i c k\,(-i\hbar)\right](2\pi)^3\, \delta^3(k-k') =$$
$$= (2\pi)^3\, \delta^3(k-k')$$

$$\left[\alpha(k), \beta^{\dagger}(k')\right] = 0$$

$$\left[\alpha(k), \beta(k')\right] = \frac{1}{8\pi\hbar c k}\left[4\pi i c k\,(i\hbar) + 4\pi i c k\,(-i\hbar)\right] = 0.$$

NOTE: we have used

$$\left[\Pi(k), A^\dagger(k')\right] = \left(\left[A(k'), \Pi^\dagger(k)\right]\right)^* = (i\hbar)^* (2\pi)^3 \delta^3(k-k') =$$
$$= -i\hbar (2\pi)^3 \delta^3(k-k')$$

$\Rightarrow$ the set $\{\alpha(k), \beta(k), \alpha^\dagger(k), \beta^\dagger(k)\}$ satisfies

Bosonic commutation rules as creation/destruction operators of harmonic oscillators

NOTE: k is still in half-space.

but naive extension $\alpha(-k) = \sqrt{\frac{1}{8\pi\hbar c k}}\left[k A(-k) + 4\pi i c \Pi(-k)\right] =$

$$= \sqrt{\frac{1}{8\pi\hbar c k}}\left[k A^\dagger(k) + 4\pi i c \Pi^\dagger(k)\right] =$$

$$= \beta(k)$$

Free-field Hamiltonian

$$H_F = \frac{1}{4\pi}\int \frac{d^3k}{(2\pi)^3}\left[c^2 |4\pi \Pi(k)|^2 + k^2 |A(k)|^2\right]$$

can be rewritten as:

$$= \int \frac{d^3k}{(2\pi)^3} \frac{\hbar c k}{2}\left[\alpha^\dagger(k)\alpha(k) + \alpha(k)\alpha^\dagger(k) + \right.$$
$$\left. + \beta^\dagger(k)\beta(k) + \beta(k)\beta^\dagger(k)\right]$$

where $\int$ is still on half-space!

Field operator

$$A(r) = \int \frac{d^3k}{(2\pi)^3} A(k)\, e^{ikr} = \int \frac{d^3k}{(2\pi)^3} \left(A(k)\, e^{ikr} + A(-k)\, e^{-ikr} \right) =$$

$$= \int \frac{d^3k}{(2\pi)^3} \left(A(k)\, e^{ikr} + A^{+}(k)\, e^{-ikr} \right) = \ldots$$

$$A(k) = \sqrt{\frac{2\pi\hbar c}{k}} \left(\alpha(k) + \beta^{+}(k) \right)$$

$$\ldots = \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{2\pi\hbar c}{k}} \left[\left(\alpha(k) + \beta^{+}(k) \right) e^{ikr} + \right.$$

$$\left. + \left(\alpha^{+}(k) + \beta(k) \right) e^{-ikr} \right] = \ldots$$

Extending definition $\alpha(-k) = \beta(k)$

$$\ldots = \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{2\pi\hbar c}{k}} \left(\alpha(k)\, e^{ikr} + \alpha^{+}(k)\, e^{-ikr} \right) = A(r)$$

With the same definition :

$$H_F = \int \frac{d^3k}{(2\pi)^3} \frac{\hbar c k}{2} \left[\alpha(k)\, \alpha^{+}(k) + \alpha^{+}(k)\, \alpha(k) \right]$$

<u>NOTE</u> : so far, we have always "forgotten" polarization degrees of freedom.

$$H_F = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon \perp k} \frac{\hbar c k}{2} \left[\alpha(k,\varepsilon)\, \alpha^{+}(k,\varepsilon) + \alpha^{+}(k,\varepsilon)\, \alpha(k,\varepsilon) \right]$$

$$A(r) = \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\hbar}{2\pi c k}} \cdot \hat{\varepsilon} \left[\alpha(k,\varepsilon)\, e^{ikr} + \alpha^{+}(k,\varepsilon)\, e^{-ikr} \right]$$

with $\hat{\varepsilon}$ unit vector $\perp \underline{k}$

Ground state

$|g\rangle$ is defined as $\alpha(k)|g\rangle = 0$ for all k's.

NOTE: proof of existence of $|g\rangle$ can be found in
C. Cohen-Tannoudji, B. Diu, F. Laloë, Quantum Mechanics,
Vol. 1, chap. 5

All excited eigenstates characterized by n_k:

$$|e\rangle = \prod_k (\alpha(k))^{n_k} |g\rangle$$

$\hookrightarrow$ mode k contains n_k photons!

Observables

$$B(r) = \nabla \times A(r) = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon \perp k} i\sqrt{\frac{2\pi\hbar c}{k}} \, (k \times \hat{\varepsilon}) \left[\alpha(k,\varepsilon) e^{ikr} - \alpha^+(k,\varepsilon) e^{-ikr} \right]$$

$$E_{\perp} = -\frac{1}{c}\dot{A} = -\frac{1}{i\hbar c}[A, H] = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \sqrt{\frac{2\pi\hbar c}{k}} \, \frac{-1}{i\cancel{\hbar c}} \, \cancel{\hbar} ck \cdot \hat{\varepsilon} \cdot \left[\alpha(k,\varepsilon) e^{ikr} - \alpha^+(k,\varepsilon) e^{-ikr} \right]$$

$$= \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} i\sqrt{2\pi\hbar c k} \, \hat{\varepsilon} \left[\alpha(k,\varepsilon) e^{ikr} - \alpha^+(k,\varepsilon) e^{-ikr} \right]$$

NOTE: light-matter interaction term $\frac{1}{2m}(P - \frac{q}{c}A)^2$ does not contribute to $\dot{A}$, nor to E.

Commutation rules

$$[A_i(r), A_j(r')] = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \sum_{\varepsilon, \varepsilon'} \frac{2\pi\hbar c}{\sqrt{kk'}} \varepsilon_i \varepsilon_j$$

$$\cdot \left[\alpha(k) e^{ikr} + \alpha^{+}(k) e^{-ikr}, \alpha(k') e^{ik'r'} + \alpha^{+}(k') e^{-ik'r'} \right] =$$

$$= \iint \sum_{\varepsilon\varepsilon'} \frac{2\pi\hbar c}{\sqrt{kk'}} \left[\delta(k-k') \delta_{\varepsilon\varepsilon'} e^{ik(r-r')} - \delta_{\varepsilon\varepsilon'} \delta(k-k') e^{-ik(r-r')} \right] \cdot \varepsilon_i \varepsilon_j$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{2\pi\hbar c}{k} P_{\perp}(k) \underbrace{\left(e^{ik(r-r')} - e^{-ik(r-r')} \right)}_{\text{odd function in } k} = 0$$

$$[E_i(r), A_j(r')] = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \sum_{\varepsilon\varepsilon'} i\sqrt{2\pi\hbar c k} \sqrt{\frac{2\pi\hbar c}{k}}$$

$$\cdot \varepsilon_i \varepsilon'_j \left[\alpha(k,\varepsilon) e^{ikr} - \alpha^{+}(k,\varepsilon) e^{-ikr}, \alpha(k',\varepsilon') e^{ik'r'} + \alpha^{+}(k',\varepsilon') e^{-ik'r'} \right]$$

$$= i\, 2\pi\hbar c \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \varepsilon_i \varepsilon_j \left[e^{ik(r-r')} + e^{-ik(r'-r')} \right]$$

$$= 4\pi i \hbar c \, \delta^{\perp}_{ij}(r-r')$$

$$[E_i(r), E_j(r')] = 0$$

Properties of the vacuum state :

zero-point (quantum) fluctuations

ground state $|vac\rangle$: $a(k,\varepsilon)|vac\rangle = 0$.

$\langle vac| E(r) |vac\rangle = 0$.

but $\langle vac| E(r)^2 |vac\rangle = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \sum_{\varepsilon\varepsilon'}$

$$(-) 2\pi\hbar c \sqrt{kk'} \; \hat\varepsilon\cdot\hat\varepsilon' \; \langle vac| \left(a(k,\varepsilon) - \cancel{a^+(k,\varepsilon)}\right)$$

$$\cdot \left(\cancel{a(k',\varepsilon')} - a^+(k',\varepsilon')\right) |vac\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} 2\pi\hbar c k = \infty$$

variance of zero-point fluctuations diverges UV

Regularized form: $\bar E(r) = \int d^3r' \, g(r-r') E(r')$.

with $g(r)$ localized around 0, but regular

$$\langle vac| \bar E(0)^2 |vac\rangle = \int d^3r \, d^3r' \, g(r) g(r') \cdot \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \cancel{\int \frac{d^3k'}{(2\pi)^3} \sum_{\varepsilon'}} .$$

$$2\pi\hbar c k \; e^{ikr} e^{-ik'r'} \; \cancel{(2\pi)^3 \delta(k-k') \delta_{\varepsilon\varepsilon'}} \; (\hat\varepsilon\cdot\hat\varepsilon')$$

$$= \int \frac{d^3k}{(2\pi)^3} 2\pi\hbar c k \int d^3r \, g(r) e^{ikr} \int d^3r' \, g(r') e^{-ikr}$$

$$= \int \frac{d^3k}{(2\pi)^3} 4\pi\hbar c k \, |\tilde g(k)|^2$$

→ the $\tilde{g}(k)$ function regularises the integral and avoids UV divergence

Similar strategy can be used to regularise light-matter Hamiltonian:

$$H_{int} = -\frac{q}{m} P \cdot \bar{A}(0) = -\frac{q}{m} \vec{P} \int \frac{d^3k}{(2\pi)^3} \cdot \hat{\varepsilon} \left(a(k,\varepsilon) e^{ikr} + a^\dagger(k,\varepsilon) e^{-ikr} \right) \sqrt{\frac{2\pi\hbar c}{k}} \cdot \tilde{g}(k)$$

where we have defined $\bar{A}(0) = \int d^3r \, g(r) \, A(r)$

Quantum fluctuations of E, B, A fields are analogous to zero-point fluctuations of quantum harmonic oscill:

$$X = \sqrt{\frac{m\omega_0}{2\hbar}} (a + a^\dagger) \quad , \quad P = \sqrt{\frac{m\hbar\omega_0}{2}} \, i(a^\dagger - a)$$

$$\langle g|X|g\rangle = 0 \quad , \quad \langle g|P|g\rangle = 0 \quad , \quad \langle g|P^2|g\rangle = \frac{m\hbar\omega}{2} \langle g| i(a^\dagger - a) \, i(a^\dagger - a)|g\rangle = \frac{m\hbar\omega_0}{2}$$

$$\langle g|X^2|g\rangle = \frac{\hbar}{2m\omega_0} \langle g|(a + a^\dagger)(a + a^\dagger)|g\rangle = \frac{\hbar}{2m\omega_0}$$

Zero-point energy $\frac{\hbar\omega_0}{2}$ comes from:

$$\langle g|H|g\rangle = \langle g| \frac{P^2}{2m} + \frac{m\omega_0^2}{2} X^2 |g\rangle = \frac{\hbar\omega_0}{2}$$

$$H_F = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \frac{\hbar c k}{2} \left(a(k,\varepsilon)\, a^{+}(k,\varepsilon) + a^{+}(k,\varepsilon)\, a(k,\varepsilon) \right)$$

$$= \underbrace{\int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \frac{\hbar c k}{2}}_{\text{zero-point energy } E_0} + \underbrace{\int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \hbar c k \; a^{+}(k,\varepsilon)\, a(k,\varepsilon)}_{\text{excitations above } |vac\rangle}$$

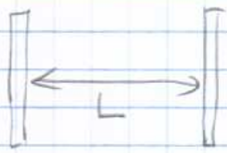
→ E_0 is UV-divergent

→ has no direct physical effect.

How to reveal <u>quantum fluctuations</u> in vacuum state:

1 - spontaneous emission is sometimes considered as a consequence of z.p.f

2 - (static) Casimir effect:



- Plane mirrors modify spectrum of z.p.f

- $E_0(L)$ is still UV-divergent, but the derivative $\frac{dE_0}{dL}$ is finite ⟶ finite force $F = -\frac{dE_0}{dL}$

⟹ CASIMIR force

3 — Dynamical Casimir effect (DCE)

- fast motion of mirrors

- z.p.f have no time to (adiabatically) follow the motion of mirrors and remain in ground state

- photons are emitted even in the absence of net currents

see e.g. :

- M. Kardar and R. Golestanian Rev. Mod. Phys. 71, 1233 ('99)

- A. Lambrecht, J. Opt. B: Quant. Semiclass. Opt. 7, S3 ('05)