**The wavefunction of a spontaneously emitted photon**

Consider a two-level atom located at position $\mathbf{r}_0$ and coupled to a scalar field by the interaction Hamiltonian

$$H_{\text{int}} = \sum_{\mathbf{k}} [\bar{\mathcal{E}}_k e^{i\mathbf{k}\mathbf{r}_0} \hat{a}_{\mathbf{k}} + \bar{\mathcal{E}}_k e^{-i\mathbf{k}\mathbf{r}_0} \hat{a}^\dagger_{\mathbf{k}}][d_{eg}\sigma^+ + d_{eg}\sigma^-]. \tag{1}$$

$\bar{\mathcal{E}}_k = \mathcal{E}_k/\sqrt{V}$ is the field amplitude for the mode of wavevector $k$ and the sum over modes is to be taken within a quantization box of volume $V$ and periodic boundary conditions. The frequency of the mode at $k$ is $\omega_k$ and the natural excitation frequency of the atom is $\omega_0$. $\sigma^+$ and $\sigma^-$ are the raising and lowering operators for the two-level atom. Both $\mathcal{E}_k$ and $d_{eg}$ are real and positive.

1. At the initial time $t = 0$, the atom is in its excited state $e$ and the field is in its vacuum state. Show that within the Rotating Wave Approximation, the state of the system at a generic time $t$ can be written in the form

$$|\psi(t)\rangle = e^{-i\omega_0 t} \left[ c_e(t)\, |e\rangle \otimes |\text{vac}\rangle + \sum_{\mathbf{k}} c_{\mathbf{k}}(t)\, |g\rangle \otimes |1:\mathbf{k}\rangle \right] \tag{2}$$

   where the one-photon states are defined as $|1:\mathbf{k}\rangle = \hat{a}^\dagger_{\mathbf{k}}|\text{vac}\rangle$. Write the equation of motion for the $c_e(t)$ and $c_{\mathbf{k}}(t)$ amplitudes.

2. Show that the field amplitude $c_{\mathbf{k}}(t)$ can be written in terms of the atomic one at previous times as:

$$c_{\mathbf{k}}(t) = \int_0^t dt'\, e^{-i(\omega_k-\omega_0)(t-t')} \left[-i d_{eg} \bar{\mathcal{E}}_k e^{-i\mathbf{k}\mathbf{r}_0}\, c_e(t')\right] \tag{3}$$

3. Inserting this expression into the equation of motion for $c_e(t)$ and exchanging the sum and the integral, write an integro-differential evolution equation for the evolution of $c_e(t)$ in the form

$$\frac{dc_e(t)}{dt} = -\int_{-\infty}^{t} dt'\, \Gamma(t-t')\, c_e(t') \tag{4}$$

   with a memory kernel

$$\Gamma(t-t') = \sum_{\mathbf{k}} (d_{eg}\bar{\mathcal{E}}_k)^2\, e^{-i(\omega_k-\omega_0)(t-t')}. \tag{5}$$

4. Assuming the frequency distribution of the field modes to be almost flat, show that the memory kernel can be approximated as

$$\Gamma(t-t') \simeq \Gamma_e\, \delta(t-t') \tag{6}$$

   with $\Gamma_e$ equal to the decay rated predicted by the Fermi golden rule. What is the time-evolution of $c_e(t)$ ?

5. Using the integral relation (3), give an expression for the field amplitude $c_{\mathbf{k}}(t)$ at late times $\Gamma_e t \gg 1$.

6. Restricting our attention to one-photon states, we can define a photon wavefunction in terms of the field operator

$$\hat{a}(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} \, \hat{a}_{\mathbf{k}} \tag{7}$$

as

$$\phi(\mathbf{r}) = \langle \text{vac} | \hat{a}(\mathbf{r}) | \psi \rangle. \tag{8}$$

Within a simplified one-dimensional model with photon dispersion $\omega_k = ck$, write an expression for the photon wavefunction at late times $\Gamma_e t \gg 1$ and give a physical interpretation of the result.

## The wavefunction of a spontaneously emitted photon



Model Hamiltonian (neglecting polarization):

$$H_{int} = \int \frac{d^3k}{(2\pi)^3} \left[ \mathcal{E}_k e^{ikr_0} \hat{a}_k + \mathcal{E}_k e^{-ikr_0} \hat{a}_k^\dagger \right] \left[ \sigma^+ + \sigma^- \right] d_{eg}$$

$$\mathcal{E}_k, d_{eg} \in \mathbb{R}, \qquad \sigma^+ = |e\rangle\langle g|, \quad \sigma^- = (\sigma^+)^\dagger = |g\rangle\langle e|$$

$$H = \int \frac{d^3k}{(2\pi)^3} \hbar\omega_k a_k^\dagger a_k + \hbar\omega_0 |e\rangle\langle e| + H_{int}$$

In calculations: $\int \frac{d^3k}{(2\pi)^3} \longrightarrow \sum_k$, $\bar{\mathcal{E}}_k = \frac{1}{\sqrt{V}} \mathcal{E}_k$, RWA

At $t=0$: $|\psi(t=0)\rangle = |e\rangle \otimes |vac\rangle$

At generic $t$: $|\psi(t)\rangle = c_e(t)\, |e\rangle \otimes |vac\rangle + \sum_k c_k(t)\, |g\rangle \otimes |1:k\rangle$

Time evolution:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

$$i\hbar \left[ \dot{c}_e(t) |e\rangle|vac\rangle + \sum_k \dot{c}_k(t) |g\rangle|1:k\rangle \right] =$$

$$= \hbar\omega_0 c_e(t) |e\rangle|vac\rangle + \hbar\omega_k c_k(t) |g\rangle|1:k\rangle +$$

$$+ \sum_k d_{eg} \bar{\mathcal{E}}_k e^{ikr_0} c_k(t) \; |e\rangle|vac\rangle +$$

$$+ \sum_k d_{eg} \bar{\mathcal{E}}_k e^{-ikr_0} c_e(t) \; |g\rangle|1:k\rangle$$

$$\begin{cases} \dot{c}_e = -i\omega_0 c_e - i \sum_n \frac{d_y \bar{\mathcal{E}}_n}{\hbar} e^{ikz_0} c_n \\ \dot{c}_n = -i\omega_n c_n - i \frac{d_y \bar{\mathcal{E}}_n}{\hbar} e^{-ikz_0} c_e \end{cases}$$

$\underline{t=0}$ : $c_e(0) = 1, \quad c_n(0) = 0.$

$\underline{\text{define}}$ : $\tilde{c}_e(t) = e^{i\omega_0 t} c_e(t)$

$\tilde{c}_n(t) = e^{i\omega_0 t} c_n(t)$

$$\begin{cases} \dot{\tilde{c}}_e = -i \sum_n \frac{d_y \bar{\mathcal{E}}_n}{\hbar} e^{ikz_0} \tilde{c}_n \\ \dot{\tilde{c}}_n = -i(\omega_n - \omega_0) \tilde{c}_n - i \frac{d_y \bar{\mathcal{E}}_n}{\hbar} e^{-ikz_0} \tilde{c}_e \end{cases}$$

$$\tilde{c}_n(t) = \int_0^t dt' \, e^{-i(\omega_n-\omega_0)(t-t')} \left[ -i \frac{d_y \bar{\mathcal{E}}_n}{\hbar} e^{-ikz_0} \tilde{c}_e(t') \right]$$

$$\dot{\tilde{c}}_e(t) = -i \sum_n \frac{d_y \bar{\mathcal{E}}_n}{\hbar} \cancel{e^{ikz_0}} \int_0^t dt' \, e^{-i(\omega_n-\omega_0)(t-t')} \left[ -i \frac{d_y \bar{\mathcal{E}}_n}{\hbar} \cancel{e^{-ikz_0}} \tilde{c}_e(t') \right]$$

$$= -\int_0^t dt' \sum_n \left( \frac{d_y \bar{\mathcal{E}}_n}{\hbar} \right)^2 e^{-i(\omega_n-\omega_0)(t-t')} \tilde{c}_e(t')$$

$$\simeq -\int_{-\infty}^t dt' \underbrace{\left[ \sum_n \left( \frac{d_y \bar{\mathcal{E}}_n}{\hbar} \right)^2 e^{-i(\omega_n-\omega_0)(t-t')} \right]}_{\tilde{\Gamma}(t-t')} \tilde{c}_e(t')$$

* integral lower bound $= -\infty$ assuming that $\tilde{\Gamma} \neq$ only for relatively short $t-t'$

$$\tilde{\Gamma}(t-t') = \int \frac{d^3k}{(2\pi)^3} \cdot \left(\frac{d_{eg}\mathcal{E}_k}{\hbar}\right)^2 e^{-i(\omega_k-\omega_0)(t-t')}$$

$$= \int dE \, \rho(E) \left(\frac{d_{eg}\mathcal{E}_k}{\hbar}\right)^2 e^{-i(\omega-\omega_0)(t-t')}$$

density of states $\quad \rho(E) = \int \frac{d^3k}{(2\pi)^3} \cdot \delta(E - \hbar\omega_k)$

As $\tilde{c}_e(t)$ is slowly varying $\longrightarrow$ component at $\omega_0$ dominates

$$\Gamma(t-t') \simeq \int d\omega \, \rho(E_0) \frac{(d_{eg}\mathcal{E}_k)^2}{\hbar} e^{-i(\omega-\omega_0)(t-t')} =$$

$$= (2\pi)\, \rho(E_0) \frac{(d_{eg}\mathcal{E}_{k_0})^2}{\hbar} \delta(t-t')$$

$$\Rightarrow \quad \dot{\tilde{c}}_e(t) = -\int_{-\infty}^{t} dt' \, (2\pi)\, \rho(E_0) \frac{(d_{eg}\mathcal{E}_{k_0})^2}{\hbar} \delta(t-t') \, \tilde{c}_e(t')$$

$$\simeq -\frac{1}{2} (2\pi)\, \rho(E_0) \frac{(d_{eg}\mathcal{E}_{k_0})^2}{\hbar} \tilde{c}_e(t)$$

$$\int_{-\infty}^{0} dt \, f(t)\, \delta(t) \simeq \frac{1}{2} f(0)$$

Effective decay rate: $\quad \Gamma_e = \frac{2\pi}{\hbar} \rho(E_0) \left(d_{eg}\mathcal{E}_{k_0}\right)^2$

Fermi golden rule

$$\Rightarrow \quad \tilde{c}_e(t) = \exp\left[-\Gamma_e t/2\right]$$

$$\tilde{c}_k(t) = \int_0^t dt' \, e^{-i(\omega_k-\omega_0)(t-t')} \left[ -i \frac{d_{eg} \bar{\mathcal{E}}_k}{\hbar} e^{-ikr_0} \right] e^{-\Gamma_e t'/2}$$

$$= e^{-i(\omega_k-\omega_0)t} \cdot \int_0^t dt' \, e^{(i(\omega_k-\omega_0) - \Gamma_e/2)t'} \cdot \left[ -i \frac{d_{eg} \bar{\mathcal{E}}_k}{\hbar} e^{-ikr_0} \right]$$

$$= e^{-i(\omega_k-\omega_0)t} \left[ -i \frac{d_{eg} \bar{\mathcal{E}}_k}{\hbar} e^{-ikr_0} \right] \frac{e^{(i(\omega_k-\omega_0) - \frac{\Gamma_e}{2})t} - 1}{i(\omega_k-\omega_0) - \Gamma_e/2} .$$

At long times $t \to \infty$:

$$\tilde{c}_k(t \to \infty) \simeq e^{-i(\omega_k-\omega_0)t} \; i \frac{d_{eg} \bar{\mathcal{E}}_k}{\hbar} e^{-ikr_0} \frac{1}{i(\omega_k-\omega_0) - \Gamma_e/2}$$

$$c_k(t \to \infty) \simeq e^{-i\omega_k t} \frac{\frac{d_{eg} \bar{\mathcal{E}}_k}{\hbar} e^{-ikr_0}}{\omega_k - \omega_0 + i\Gamma_e/2}$$

Photon wavefunction in scalar model:

$$\psi(r) = \langle vac | \, a(r) \, | \psi \rangle =$$

$$= \langle vac | \sum_k \frac{1}{\sqrt{V}} e^{ikr} \, a_k | \psi \rangle =$$

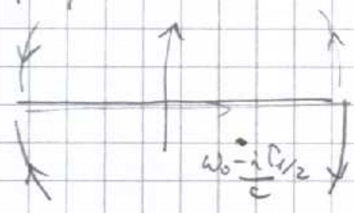
$$= \frac{1}{\sqrt{V}} \sum_k e^{ikr} \, e^{-i\omega_k t} \frac{d_{eg} \bar{\mathcal{E}}_k/\hbar \, e^{-ikr_0}}{\omega_k - \omega_0 + i\Gamma_e/2} =$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{ik(r-r_0)} e^{-i\omega_k t} \frac{d_{eg} \mathcal{E}_k}{\hbar} \frac{1}{\omega_k - \omega_0 + i\Gamma_e/2}$$

For simplicity : 1D model , $\omega_k = c \cdot k$

$$\phi(x) = \int \frac{dk}{2\pi}\, e^{ik[(x-x_0)-ct]} \, \frac{d_y \mathcal{E}_k/\hbar}{ck - \omega_0 + i\Gamma_e/2}$$

for $x - x_0 - ct < 0$ → close in lower half-plane.

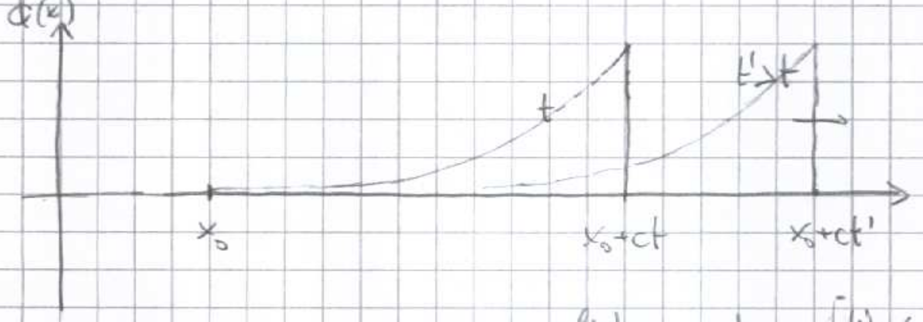



$$= -i \frac{d_y \mathcal{E}_{k_0}}{\hbar c}\, e^{+\frac{i}{c}(\omega_0 - i\frac{\Gamma_e}{2})(x - x_0 - ct)}$$

for $x - x_0 - ct > 0$ → close in upper half-plane

$$= \emptyset$$

$$\phi(x) = -\frac{i}{c\hbar}\, d_y \mathcal{E}_{k_0}\, e^{i(\omega_0 - i\Gamma_e/2)(\frac{x-x_0}{c} - t)} \quad \Theta\big(ct - (x - x_0)\big)$$




+ oscillations at $\left[\frac{\omega_0}{c}(x - x_0) - \omega_0 t\right]$

## Single photon vs coherent state

At long times $t \to \infty$ :

$$c_e(t) \longrightarrow 0$$

$$c_k(t) \longrightarrow e^{-i\omega_k t} \quad \frac{d_{eg}\mathcal{E}_k}{\hbar} e^{-ikx_0} \frac{1}{\omega_k - \omega_0 + i\Gamma_e/2}$$

k-space photon wavefunction

in real-space $\quad \phi(x,t) = -\frac{i}{c\hbar} d_{eg} \mathcal{E}_{k_0} e^{i(\omega_0 - i\Gamma_e/2)\left(\frac{x-x_0}{c} - t\right)} \times$

$$\times\, \vartheta(ct - (x - x_0))$$

final state $\quad |\psi_f\rangle = \int dx \; \phi(x,t) \; \hat{\phi}^+(x) \; |vac\rangle$

↳ eigenstate of photon number operator

$$\hat{N} = \int dx \; \hat{\phi}^+(x) \hat{\phi}(x)$$

$$\hat{N}|\psi_f\rangle = 1 \cdot |\psi_f\rangle$$

Time-dependent current : $\quad j(r,t) = j_0(r) \cdot \tilde{j}_T(t) \, e^{-i\omega_0 t}$

↳ $\mathbb{C}$-number

Assume : $\begin{cases} j_0(r) \text{ localized in space on small size } \sigma \\ \tilde{j}_T(t) \text{ slowly varying envelope of duration } T \end{cases}$

$$\left.\begin{aligned} j_0(r) &= j_0\, e^{-r^2/2\sigma^2}, \quad \sigma \ll c/\omega_0 \\ \tilde{j}_T(t) &= e^{-t^2/2T^2} \end{aligned}\right\} \text{example}$$

field in coherent state with amplitude $\alpha_k$:

$$\hat{a}_k |\psi\rangle = \alpha_k |\psi\rangle$$

$$\alpha_k = - \int_{-\infty}^{t} dt' \; \bar{j}_k(t')\, e^{-ick(t-t')}$$

$$\text{with } \bar{j}_k(t) = \sqrt{\frac{2\pi\hbar}{ck}} \int d^3r \; j(r,t)\, e^{ikr}$$

$\hookrightarrow$ neglected polarization.

$$\alpha_k(t\to\infty) = - \int_{-\infty}^{\infty} dt' \sqrt{\frac{2\pi\hbar}{ck}} \cdot \underbrace{\int d^3r \; j_0(r)\, e^{ikr}}_{(2\pi)^{3/2}\sigma^3 j_0} \cdot \tilde{j}_T(t')\, e^{-i\omega_0 t'} \cdot \times e^{-ick(t-t')}$$

$$= - \sqrt{\frac{2\pi\hbar}{ck}} \cdot (2\pi)^{3/2} j_0 \sigma^3 \cdot \int_{-\infty}^{\infty} dt' \; e^{-t'^2/2T^2}\, e^{-i(\omega_0-ck)t'} \; e^{-ickt}$$

$$= - \sqrt{\frac{\hbar}{ck}}\, (2\pi)^2 j_0 \sigma^3 \cdot T \cdot \sqrt{2\pi}\; e^{-(\omega_0-ck)^2 T^2/2}\; e^{-ickt}$$

In real space:

$$\alpha(r,t) = \int \frac{d^3k}{(2\pi)^3}\, \alpha_k(t)\, e^{ikr} =$$

$$= \int \frac{d^3k}{(2\pi)^3} \left[ -\sqrt{\frac{\hbar}{ck}}\, (2\pi)^2\, j_0\, \sigma^3\, T \sqrt{2\pi}\, e^{-ickt} \cdot e^{-(\omega_0 - ck)^2 T^2/2} \right] e^{ikr}$$

1-D model: $\int dr\, j_0(r)\, e^{ikr} = \sqrt{2\pi}\, \sigma\, j_0$

$$\alpha(r,t) = \int \frac{dk}{2\pi}\, e^{ikr} \left[ -\sqrt{\frac{\hbar}{ck}}\, (2\pi)\, j_0\, \sigma \cdot T \sqrt{2\pi} \cdot e^{-ickt} \cdot e^{-(\omega_0 - ck)^2 T^2/2} \right]$$

$$= -\sqrt{\frac{2\pi\hbar}{\omega_0}}\, j_0\, \sigma\, T \int dk\, e^{ik(r-ct)}\, e^{-(\omega_0 - ck)^2 T^2/2}$$

$$= -\sqrt{\frac{\hbar}{\omega_0}}\, \frac{2\pi j_0 \sigma}{c}\, e^{-i\omega_0 t}\, e^{-(r-ct)^2/2c^2T^2}$$

↳ propagating wavepacket with Gaussian shape

State of field : coherent state

$$|\alpha\rangle = \prod_n e^{-|\alpha_n|^2/2}\, e^{\alpha_n \hat{a}_n^+}\, |vac\rangle$$

$$= \exp\left[-\frac{1}{2}\sum_n |\alpha_n|^2 + \sum_n \alpha_n \hat{a}_n^+\right] |vac\rangle$$

$$= \exp\left[-\frac{1}{2}\int d^3r\, |\alpha(r)|^2 + \int d^3r\, \alpha(r)\hat{\psi}^+(r)\right] |vac\rangle$$

$$= |coh : \alpha(r)\rangle \quad \text{such that}$$

$$\hat{\psi}(r)\, |coh : \alpha(r)\rangle = \alpha(r)\, |coh : \alpha(r)\rangle$$

$$\langle coh : \alpha(r) | \hat{N} | coh : \alpha(r)\rangle = \langle coh : \alpha(r)| \int d^3r\, \hat{\psi}^+(r)\psi(r)\, |coh : \alpha(r)\rangle =$$

$$= \int d^3r\, |\alpha(r)|^2 = N$$

$$\langle coh : \alpha(r) | \hat{N}^2 | coh : \alpha(r)\rangle = N^2 + N$$

$$\Delta N^2 = \langle N^2\rangle - \langle N\rangle^2 = N \neq 0$$ has some number fluctuations.

Many photons with same wavefunction $\alpha(r)$. $\left[\hat{a}_\alpha^+ = \int d^3r\, \alpha(r)\hat{\psi}^+(r)\right]$

\# of photons fluctuates.