

What is entanglement?

Bell inequalities & Aspect's experiment.

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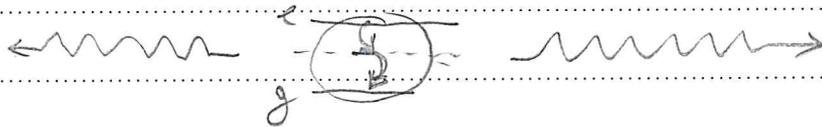
Atom emits entangled pair of photons:

$$| \psi \rangle = \frac{1}{\sqrt{2}} (| \sigma_+ \sigma_- \rangle + | \sigma_- \sigma_+ \rangle) = \frac{1}{\sqrt{2}} (| hh \rangle + | vv \rangle)$$

left propagating

right propagating

$$= \frac{1}{\sqrt{2}} (| \nearrow \nearrow \rangle + | \searrow \searrow \rangle)$$



- L measures $\sigma_x \Rightarrow$ R sees σ_x
- L measures $\sigma_z \Rightarrow$ R sees σ_z

One is tempted to interpret "classically":

$$h = 1/2 \longrightarrow (\sigma_+, \sigma_-) ; \quad v = 1/2 \longrightarrow (\sigma_-, \sigma_+)$$

What about measuring in h, v basis?



$$\begin{array}{l}
 \sigma_+ \text{ photon} \rightarrow \left. \begin{array}{l} \left. \begin{array}{l} p=1/2 \rightarrow h \\ p=1/2 \rightarrow v \end{array} \right\} \right\} \\
 \sigma_- \text{ photon} \rightarrow \text{the same.} \end{array} \right\} p(h, v | \sigma_{\pm}) = 1/2.
 \end{array}$$

Elementary probability theory:

locality assumption

$$\begin{aligned}
 p(h, h) &= \sum_{\substack{\sigma_+, \sigma_- \\ \text{complete set} \\ \text{of states}}} p(\sigma_{\pm}, \sigma_{\pm}) \cdot \overbrace{p(h|\sigma_+)}^L \cdot \overbrace{p(h|\sigma_-)}^R \\
 &= \frac{1}{2} p(h|\sigma_+) p(h|\sigma_-) + \frac{1}{2} p(h|\sigma_-) p(h|\sigma_+) \\
 &= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}
 \end{aligned}$$

The same for all other combinations:

	h	v
h	1/4	1/4
v	1/4	1/4

while

	σ_+	σ_-
σ_+	0	1/2
σ_-	1/2	0



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Using QM state $|k\rangle = \frac{|hh\rangle + |vv\rangle}{\sqrt{2}}$

(which is rewriting of $\frac{|0_+0_-\rangle + |0_-0_+\rangle}{\sqrt{2}}$

in another basis), one finds:

	h	v
h	$1/2$	0
v	0	$1/2$

and expt confirms!

So, classical probability theory does not hold in QM!

Spin-1/2, singlet state $|k\rangle = \frac{1}{\sqrt{2}} [|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$

$J=0$ representation of $SO_3 \rightarrow$ keeps real form in all bases.

e.g. $|k\rangle = \frac{1}{\sqrt{2}} [|\leftarrow\rightarrow\rangle - |\rightarrow\leftarrow\rangle]$

Note that photons are spin-1, and not 1/2.

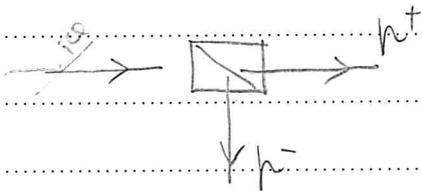


More generally: how to construct "local hidden variable theories"?

λ = "hidden variable". Work in linear polarisation basis

$$\left\{ \begin{array}{l} n_{\theta}^{+}(\lambda) = \text{probability of having a photon through polariser @ } \theta \text{ given } \lambda \\ n_{\theta}^{-}(\lambda) = \text{probability of having a photon reflected at polariser @ } \theta \text{ given } \lambda \end{array} \right.$$

$$n_{\theta}^{+}(\lambda) + n_{\theta}^{-}(\lambda) = 1$$



$$P_{\theta_1, \theta_2}^{\pm, \pm} = \int d\lambda \rho(\lambda) \cdot n_{\theta_1}^{\pm}(\lambda) \cdot n_{\theta_2}^{\pm}(\lambda)$$

distribution of λ

L

R

each probability independent



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$$E(\theta_1, \theta_2) = \frac{\langle (I_1^+ - I_1^-)(I_2^+ - I_2^-) \rangle}{\langle (I_1^+ + I_1^-)(I_2^+ + I_2^-) \rangle} \Big|_{\theta_1, \theta_2}$$

$$= P_{\theta_1 \theta_2}^{++} + P_{\theta_1 \theta_2}^{--} - P_{\theta_1 \theta_2}^{+-} - P_{\theta_1 \theta_2}^{-+} =$$

$$= \int d\lambda e(\lambda) \underbrace{(n_{\theta_1}^+(\lambda) - n_{\theta_1}^-(\lambda))}_{S_{\theta_1}(\lambda)} \underbrace{(n_{\theta_2}^+(\lambda) - n_{\theta_2}^-(\lambda))}_{S_{\theta_2}(\lambda)}$$

with $|S_{\theta}(\lambda)| \leq 1$

$$E(\theta_1, \theta_2) - E(\theta_1, \theta_2') = \int d\lambda e(\lambda) \cdot [S_{\theta_1}(\lambda) S_{\theta_2}(\lambda) + S_{\theta_1}(\lambda) S_{\theta_2'}(\lambda)] =$$

$$= \int d\lambda e(\lambda) [S_{\theta_1}(\lambda) S_{\theta_2}(\lambda) \pm S_{\theta_1}(\lambda) S_{\theta_2}(\lambda) S_{\theta_1'}(\lambda) S_{\theta_2'}(\lambda) - S_{\theta_1}(\lambda) S_{\theta_2'}(\lambda) \mp S_{\theta_1}(\lambda) S_{\theta_2}(\lambda) S_{\theta_1'}(\lambda) S_{\theta_2}(\lambda)]$$

$$= \int d\lambda e(\lambda) \left\{ S_{\theta_1}(\lambda) S_{\theta_2}(\lambda) [1 \pm S_{\theta_1'}(\lambda) S_{\theta_2'}(\lambda)] + \right.$$

$$\left. - S_{\theta_1}(\lambda) S_{\theta_2'}(\lambda) [1 \pm S_{\theta_1'}(\lambda) S_{\theta_2}(\lambda)] \right\}$$

$|...| \leq 1$

LG

$$|E(\theta_1, \theta_2) - E(\theta_1, \theta_2')| \leq \int d\lambda \rho(\lambda)$$

$$\underbrace{[1 \pm S_{\theta_1'}(\lambda) S_{\theta_2}(\lambda)]}_{\geq 0} + \underbrace{[1 \pm S_{\theta_1'}(\lambda) S_{\theta_2'}(\lambda)]}_{\geq 0}$$

$$= 2 \pm [E(\theta_1' \theta_2') + E(\theta_1' \theta_2)]$$

We have used that

$$|\langle a - b \rangle| \leq \langle |a| \rangle + \langle |b| \rangle$$

then:

$$-2 \mp [E(\theta_1' \theta_2') + E(\theta_1' \theta_2)] \leq E(\theta_1 \theta_2) - E(\theta_1 \theta_2') \leq 2 \pm [E(\theta_1' \theta_2') + E(\theta_1' \theta_2)]$$

$$\Rightarrow |E(\theta_1 \theta_2) - E(\theta_1 \theta_2') + E(\theta_1' \theta_2') + E(\theta_1' \theta_2)| \leq 2$$

Clausen-Horne-Shimony-Holt inequality

for any local hidden variable theory



Simple proof of CHSH :

$$S = (a+a')b + (a-a')b'$$

with $a, a', b, b' \in \{-1, 1\}$

$$\Rightarrow S \in \{-2, 2\}$$

Proof: if $a=a'$ $\begin{cases} a-a'=0 \\ |a+a'|=2 \end{cases}$

$$\Rightarrow S = \underbrace{(a+a')}_{1+1=2} \cdot \underbrace{b}_{1-1=1}$$

Analogously if $a \neq a'$ $\begin{cases} a+a'=0 \\ |a-a'|=2 \end{cases}$

So $|S| \leq 2$ for any distribution $p(a, a', b, b')$

In particular, take $a =$ outcome of θ_1

$a' =$ " θ_1'

$b =$ " θ_2

$b' =$ " θ_2'

$$S = ab + a'b + ab' - a'b'$$

and note $\langle ab \rangle = E(\theta_1, \theta_2)$

QM result for $|h\rangle = \frac{1}{\sqrt{2}}(|hh\rangle + |vv\rangle)$ $E(\theta_1, \theta_2) = \cos[2(\theta_1 - \theta_2)]$

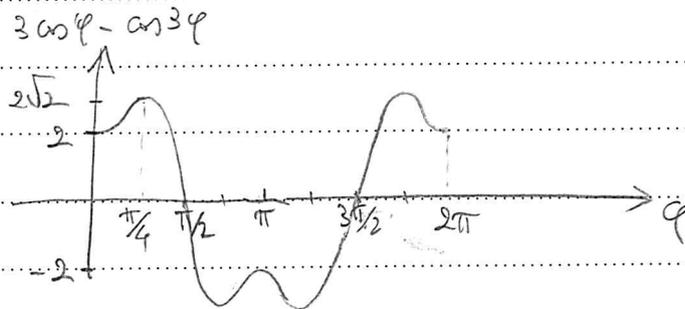
Try to break CHSH: \neq maximise $E(\theta_1, \theta_2)$
 $E(\theta_1', \theta_2')$
 $E(\theta_1', \theta_2)$

* minimize $E(\theta_1, \theta_2')$

Good choice: have $\theta_1 - \theta_2, \theta_1' - \theta_2, \theta_1' - \theta_2'$ equal
 $= \varphi/2$

E.g. for $\varphi = \pi/2 \Rightarrow 3\cos\varphi - \cos 3\varphi = 2\sqrt{2} > 2$

Maximises violation from CHSH



\Rightarrow violations exceed
by 22.5° .



Experiments:

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① Aspect, Grangier, Roger PRL 47, 460 ('81)

② " " PRL 49, 91 ('82)

③ Aspect, Dalibard, Roger PRL 49, 1804 ('82)

① → single port detection after polarizers

② → both ports measured

$$\begin{cases} S_{\text{count}} = 2.697 \pm 0.015 \\ S_{\text{QM}} = 2.70 \pm 0.05 \\ \text{CHSH bound: } |S| \leq 2 \end{cases}$$

still open loophole: quantum efficiency P.M. $\ll 1$
→ has to ensure that detected result is faithful

③ → time-varying analysers $-1 \leq \tilde{S} \leq 0$
($\tilde{S}_{\text{count}} = 0.101 \pm 0.020$ vs $\tilde{S}_{\text{QM}} = 0.112$)

switch polarizer while photons are on the way. Back-communication of polarizer state to other not allowed by causality.

(switches every $T_s \approx 10$ ns.
 $v_s \leq \frac{1}{c}$ distance between polarizers.)



④ All loopholes closed in Hensen et al, Nature 526, 682 (2015) 19

Entanglement and "no-signalling" :

no information can be transmitted by performing measurements on a sub-system.

Formally : all observables at A are determined by $\rho_A = \text{Tr}_B(\rho_{AB}) = \sum_{n=1}^{d_B} \langle b_n | \rho_{AB} | b_n \rangle$.

ρ_A is invariant under any unitary transformation modifying the state at B, that is under any physical operation performed by B. The same holds for wider set of operations involving measurements.

* No signalling can be used to impose further constraints on alternative theories.

A new logic :

$\{e_i\}$ and $\{f_j\}$ basis of \mathcal{H} corresponding to different operators, e.g. S_z and S_x .

$$P(e_i) = |a(e_i)|^2 = \left| \sum_j \langle e_i | f_j \rangle \cdot a(f_j) \right|^2$$

Can be interpreted in terms of Feynman diagrams

$$a(i \rightarrow f) = \sum_{\text{paths}} a(i \rightarrow a) a(a \rightarrow b) \dots a(z \rightarrow f)$$

$$P(i \rightarrow f) = |a(i \rightarrow f)|^2$$

Back to $\rho(h, h)$ of neg. 2:

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$$\rho(h, h) = |\alpha(h, h)|^2 \quad \text{with}$$

$$\alpha(h, h) = \sum_{\pm, \pm} \alpha(\sigma_{\pm}, \sigma_{\pm}) \cdot \langle h | \sigma_{\pm} \rangle \langle h | \sigma_{\pm} \rangle$$

$$\alpha(\sigma_{\pm}, \sigma_{\pm}) = \begin{pmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}$$

$$\langle h | \sigma_{\pm} \rangle = 1/\sqrt{2}$$

$$[\sigma_{\pm} = \frac{1}{\sqrt{2}} (|h\rangle \pm i |v\rangle)]$$

$$\langle v | \sigma_{\pm} \rangle = \pm 1/\sqrt{2}$$

$$\Rightarrow \alpha(h, h) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

and thus $\rho(h, h) = 1/2$.

And: $\alpha(h, v) = \sum_{\pm, \pm} \alpha(\sigma_{\pm}, \sigma_{\pm}) \langle h | \sigma_{\pm} \rangle \langle v | \sigma_{\pm} \rangle =$

$$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \cdot \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \cdot \left(-\frac{i}{\sqrt{2}}\right) = 0.$$

and thus $\rho(h, v) = 0$.

Bibliography

R. Feynman, "QED" (sinulgetivo su QM e logice dei diagrammi)

Wells - Millman, "Quantum Optics"
(number textbook at graduate level)

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"Multiregional gases and Quantum Information".